THIRTY-FIRST ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

PART I

October 7, 1987

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. **Check to insure that your six-digit code number has been recorded correctly.** Do not make calculations on the answer sheet. Fill in circles completely and darkly.
2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheeet.
3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.
4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer.
5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "e) none of these" or "e) none of the above." If you believe none of the first four alternatives to be correct, mark e) in such cases.
6. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes, or any other aid is prohibited. If you have questions concerning the instructions, ask them now.
7. You may now open the test booklet and begin.

THIRTY-FIRST ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

1. If $8^x = \left(\frac{1}{2}\right)^{x+1}$, then what is the value of x?

   a) $-\frac{1}{2}$ b) $-\frac{1}{4}$ c) 0 d) $\frac{1}{4}$ e) $\frac{1}{2}$

2. How many real number solutions are there to the equation $|5 - |x|| = 1$?

   a) none b) 1 c) 2 d) 3 e) more than 3

3. If $x \neq 1$, then which of the following is equal to $1 + x^2 + x^4 + x^6 + \cdots + x^{1986}$?

   a) $\frac{x^{1987} - 1}{x - 1}$ b) $\frac{x^{1987} - 1}{x^2 - 1}$ c) $\frac{x^{1988} - 1}{x - 1}$ d) $\frac{x^{1988} - 1}{x^2 - 1}$ e) none of these

4. The 15 boys in a class have a mean (arithmetic average) weight of 117 pounds, and the 20 girls in the class have a mean weight of 103 pounds. What is the mean weight of all 35 pupils in the class?

   a) 113 b) 111 c) 110 d) 109 e) 107

5. How many positive integers less than one million are divisible by neither 2 nor 5?

   a) 100,000 b) 200,000 c) 300,000 d) 400,000 e) 500,000

6. Which of the following is equal to $\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}$?

   a) $2+\sqrt{3}$ b) $2-\sqrt{3}$ c) $\sqrt{5}$ d) $\frac{1+\sqrt{5}}{2}$ e) 0

7. Which of the following numbers equals the repeating decimal $0.1\overline{3} = 0.13333\ldots$?

   a) $\frac{1}{3}$ b) $\frac{1}{9}$ c) $\frac{2}{15}$ d) $\frac{13}{99}$ e) $\frac{13}{100}$

8. Which of the following equals $\tan(\arcsin 0.2)$?

   a) $\frac{0.2}{\sqrt{0.96}}$ b) 2 c) $\sqrt{0.96}$ d) $\frac{1}{\sqrt{0.96}}$ e) $\frac{1}{\sqrt{1.04}}$

9. A small construction company has brought 80 gallons of "arctic blend" antifreeze mixture from its upper peninsula shop to its lower peninsula shop. This blend is 60% pure antifreeze and 40% water. How many gallons of water will they have to add to the mixture to dilute it to the standard strength of 50% pure antifreeze?

   a) 40 b) 16 c) 12 d) 10 e) 8

10. What is the area of the triangle bounded by the lines $y = 2x$, $y = -\frac{1}{2}x$, and $x = 4$?

a) 6 b) 12 c) 16 d) 20 e) 32

11. What is the solution of the equation $x^2 - 2x + 10 = 0$ (where $i = \sqrt{-1}$)?

a) $1 \pm 3i$ b) $-1 \pm 3i$ c) $2 \pm 3i$ d) $-2 \pm 3i$ e) 0 and 2

12. If R is a positive real number that is 1 greater than twice its reciprocal, then what is the value of R?

a) $\frac{-1+\sqrt{5}}{2}$ b) 1 c) 2 d) $\frac{1}{2}$ e) $\frac{1}{3}$

13. The number $5! = 120 = 2^3 \cdot 3 \cdot 5$ has 3 different prime factors. How many different prime factors does 20! have?

a) 7 b) 8 c) 10 d) 18 e) 19

14. Which of the following lines intersects the x-axis to the left of the origin?

a) $y - 3x = -3$ b) $2y + 3x = 0$ c) $2y - 3 = 3x$ d) $3y = -1$ e) $3x = 1$

15. Suppose the following statements are all true.

- If the Tigers play on Friday, then the Lions play on Sunday.
- If the Tigers do not play on Friday, then the Cubs play on Saturday.
- If the Bears play on Monday, then the Tigers play on Friday.

If the Cubs in fact do not play on Saturday, then which of the following statements must be true?

a) The Lions play on Sunday.
b) The Lions do not play on Sunday.
c) The Tigers do not play on Friday.
d) The Bears play on Monday.
e) The Bears do not play on Monday.

16. Diane walked 3 miles per hour from Carbondale to Cherrydale to pick up a bicycle. How fast will she have to ride back in order to average 5 miles per hour for the whole round trip?

a) 7 MPH b) 10 MPH c) 12 MPH d) 15 MPH e) it is impossible to ride fast enough

17. What is the smallest positive angle θ satisfying the equation $\tan\theta = 2\sin\theta$?

a) 15° b) 30° c) 45° d) 60° e) there is no solution

18. What are the solutions of the equation $(x^2-2)^2-3=0$?

a) only $\sqrt{2}+\sqrt{3}$

b) $\sqrt{2}+\sqrt{3}$ and others

c) only $\sqrt{2+\sqrt{3}}$

d) $\sqrt{2+\sqrt{3}}$ and others

e) four imaginary numbers

19. Consider the exponential equation $2^x+1=5^{-x}$. Which of the following statements describes the solution set?

a) There are no real solutions.

b) There are two or more real solutions.

c) There is exactly one real solution, which is positive.

d) There is exactly one real solution, namely $x=0$.

e) There is exactly one real solution, which is negative.

20. Each of three fair coins has been painted on its two sides to identify the two sides (instead of heads and tails). The first coin is red on one side and white on the other. The second coin is white on one side and black on the other. The third coin is black on one side and red on the other. All the coins are tossed into the air, and they fall side by side. What is the probability that all three colors appear face up?

a) $\frac{1}{8}$ b) $\frac{1}{6}$ c) $\frac{1}{4}$ d) $\frac{1}{3}$ e) $\frac{1}{2}$

21. A kindergarten teacher has brought three ducklings to class. During a demonstration involving the ducklings, two of the participants attain the ages of one million seconds and one billion seconds, respectively. Who or what are these most likely to be?

a) two students

b) two ducklings

c) a duckling and a student

d) a student and the teacher

e) a duckling and the teacher

22. Define the function f on positive integers greater than 1 by letting $f(n)$ be the smallest prime factor of n. What must be true about the value of $\frac{f(n)\cdot f(n+1)}{f(n(n+1))}$?

a) It must be an odd prime number.

b) It must equal 1.

c) It must equal 2.

d) It may be a composite number.

e) It may be a rational number which is not a integer.

23. If x and y are positive real numbers such that $x+y=A$ and $x^{-1}+y^{-1}=B$, then what is the value of xy?

a) AB b) $A+B$ c) AB^{-1} d) $A^{-1}B$ e) $(AB)^{-1}$

24. If a, b, and c are real numbers greater than 1 such that $a^{(b^c)}=(a^b)^c$, then what is b in terms of c?

a) $b=c^{\left(\frac{1}{c-1}\right)}$ b) $b=\frac{c}{c-1}$ c) $b=1+c^{\left(\frac{1}{c}\right)}$ d) $b=c^{(1-c)}$ e) $b=(c-1)^c$

25. If $\log x$ denotes the logarithm of x to the base 10, then what is the largest real number n such that $\log(\log n)\leq 10$?

a) 10^{10} b) 100^{10} c) 10^{20} d) 10^{100} e) $10^{(10^{10})}$

26. What is the remainder when 7^{1987} is divided by 5?

a) 4 b) 3 c) 2 d) 1 e) 0

27. What is the largest value that $-2x^2+5x+1$ attains for real numbers x?

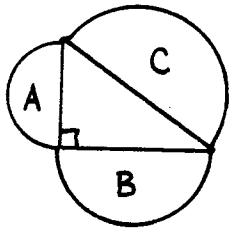
a) 1 b) $\frac{5}{4}$ c) $\frac{33}{8}$ d) $\frac{25}{16}$ e) $\frac{41}{16}$

28. Let A be the sum $1+2+\cdots+100$ of the first 100 positive integers. Let B be the sum $101+102+\cdots+200$ of the next 100 positive integers. What is the greatest common divisor of A and B?

a) 1 b) 2 c) 50 d) 100 e) 101

29. Three semicircles are constructed on the sides of a right triangle as shown. Let A, B, and C be their areas. Then which of the following statements is necessarily true?

a) $A+B=C$
b) $A+B>C$
c) $A^2+B^2=C^2$
d) $A^2+B^2>C^2$
e) none of these



30. What is the sum of the four real roots of the polynomial $4x^4-24x^3+39x^2-3x-23$?

a) $\frac{3}{4}$ b) 3 c) 6 d) 23 e) 24

31. Three players each roll a pair of ordinary six-sided dice. What is the probability that none of the players will roll a total of 7?

a) $3 \cdot \left(\frac{1}{6}\right)$ b) $3 \cdot \left(\frac{1}{7}\right)$ c) $\left(\frac{5}{6}\right)^3$ d) $\left(\frac{6}{7}\right)^3$ e) $1 - \left(\frac{1}{6}\right)^3$

32. Which of the following describes the graph of the equation $x^2 + y^2 - 2x + 4y + 6 = 0$?

a) a circle
b) an ellipse which is not a circle
c) a hyperbola
d) a pair of intersecting lines
e) the empty set

33. What is the solution set of the inequality $|x + 1| < |x - 1|$?

a) $\{x \mid -1 < x < 1\}$
b) $\{x \mid x < 0\}$
c) $\{x \mid x < 1\}$
d) $\{x \mid x > 1\}$
e) the empty set

34. Which of the following expressions is equal to $2\log_b(x + 1) + 2\log_b(x - 1)$ for all real numbers $b > 1$ and $x > 1$?

a) $2\log_b(2x)$ b) $\log_b(x^2 - 1)$ c) $\log_b((x^2 - 1)^2)$ d) $\log_b(4(x^2 - 1))$ e) $4\log_b(x^2 - 1)$

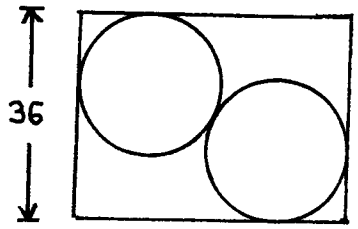
35. A tourist started the day with a few \$10 bills to spend. In every shop he entered he spent one half of his current cash, plus one dollar. At the end of the day he had spent all his money. What is the smallest number of shops he could have visited?

a) 1 b) 2 c) 3 d) 4 e) more than 4

36. What can be said about the function $f(x) = \frac{1}{1 - x^2}$?

a) It is increasing for $-1 < x < 1$.
b) It is decreasing for $-1 < x < 1$.
c) It is increasing for $-1 < x < 0$ and decreasing for $0 < x < 1$.
d) It is decreasing for $-1 < x < 0$ and increasing for $0 < x < 1$.
e) It is not defined for x such that $-1 < x < 1$.

37. In the figure, two circles of diameter 26 are tangent to each other and to two adjacent sides of the rectangle, as shown. The width of the rectangle is 36. What is its length?



a) 39
b) 48
c) 50
d) 52
e) 56

38. For what values of the constants A and B does the following system of two equations have at least one solution for x and y?

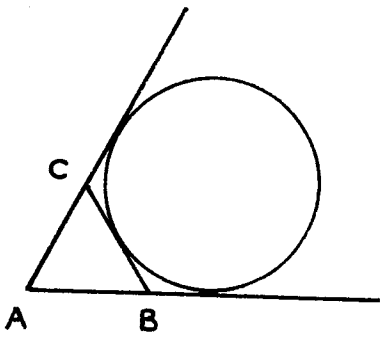
$$Ax + y = B$$
$$x + By = A$$

a) all of them
b) exactly those with $AB = 1$
c) exactly those with $AB \neq 1$
d) exactly those with either $A = B = 1$ or else $AB \neq 1$
e) exactly those with one and only one of A and B equal to 1

39. Each of five pupils—Alex, Beth, Conrad, David, and Elizabeth—is to be assigned to one of three counselors—Mr. Jones, Ms. Smith, and Dr. Wilson—so that every counselor gets either one or two pupils. In how many ways can this be done?

a) 18 b) 30 c) 60 d) 81 e) 90

40. In the figure, the sides AB and AC of the **equilateral** triangle ABC have been extended, and the circle tangent to these extended sides and to BC, as shown, has radius 4. What is the perimeter of $\triangle ABC$?



a) $4\sqrt{3}$
b) $6\sqrt{3}$
c) $8\sqrt{3}$
d) 12
e) 14

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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