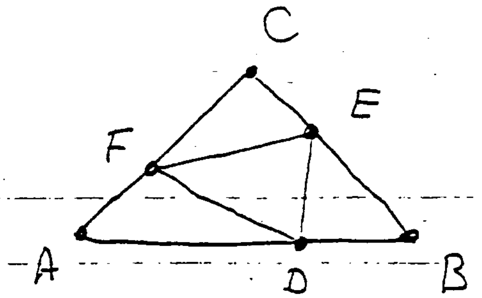
1. 

$0 < \alpha < \frac{1}{2}$

$$\frac{DB}{AB} = \frac{EC}{BC} = \frac{FA}{CA} = \alpha$$

a) Since $\triangle ABC$ is equilateral & equiangular,

$\overline{CE} = \overline{DB} = \overline{FA}$ and $\overline{BE} = \overline{CF} = \overline{AD}$

So $\triangle ADF \cong \triangle BED \cong \triangle CFE$
by side - angle - side.

Then by using corresponding parts of congruent triangles, one shows that

$\angle FED \cong \angle EDF \cong \angle DFE$

Hence, $\triangle DEF$ is equilateral.

b) Let h be the altitude of $\triangle ABC$ on side AB and h' ~~be~~ the altitude of $\triangle AFD$ on sid AD. Then by corresponding parts of similar right triangles we have

$$h' = \alpha h$$

(over)

1 b) (cont.) Now area of $\Delta AFD = \frac{\overline{AD}}{2} h'$

and area of $\Delta ABC = \frac{\overline{AB}\, h}{2}$

But $\overline{AD} = (1-\alpha)\overline{AB}$, Thus

$$\text{area of } \Delta AFD = \frac{\alpha h (1-\alpha)\overline{AB}}{2}$$

$$= \alpha(1-\alpha)(\text{area of } \Delta ABC)$$

Furthermore, since $\Delta AFD \cong \Delta FEC \cong \Delta DBE$ we have,

$$\text{area of } \Delta ABC = \text{area of } \Delta DEF + 3 \text{ area of } \Delta AF$$

$$\text{area of } \Delta ABC = \text{area of } \Delta DEF + 3\alpha(1-\alpha)(\text{area of } \Delta$$

Thus for area of $\Delta DEF = \frac{1}{2}$ area of ΔABC we would have

$$1 = \frac{1}{2} + 3\alpha(1-\alpha)$$

or $$\frac{1}{2} = 3\alpha - 3\alpha^2$$

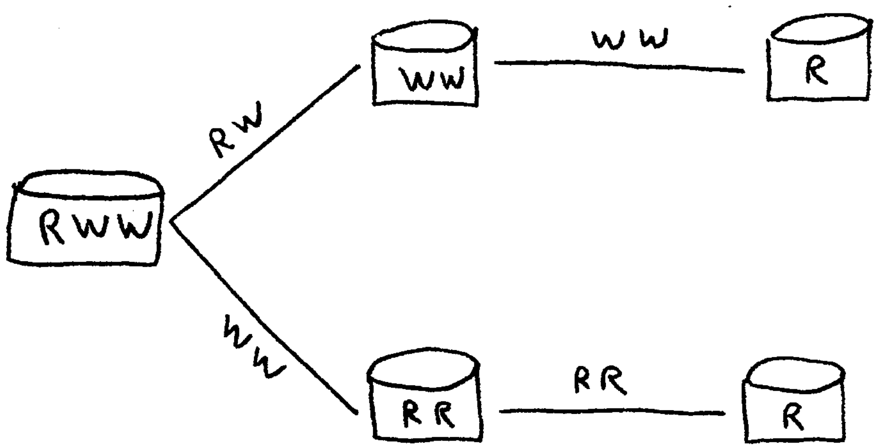
$$3\alpha^2 - 3\alpha + \frac{1}{2} = 0$$

$$\alpha = \frac{3 \pm \sqrt{3}}{6}$$

But $\alpha = \frac{3-\sqrt{3}}{6}$ is less than $\frac{1}{2}$

MMPC 1986 Guide to Grading Question 2

(a)--3 points. A red ball has to remain. An exhaustive search is probably what most contestants will use, and it's very easy, of course. In tree form it will look something like this:



Such a picture should receive full credit. It is easy to give a verbal argument as well. Give 2 points if there is a minor omission or unclarity in the answer. The correct answer (red), with no justification or with invalid justification, should perhaps receive 1 point, as should a reasonable start on a path to the solution.

(b)--7 points. The general result is simply this: *the parity of the number of white balls in the bowl is invariant under the operation.* This is clear from looking at the three possibilities, hardly requiring writing down a proof. It seems to be a case of "either you see the trick or you don't." Therefore at the end (with 1 ball left), there have to be still an even number of white balls, namely 0, so the remaining ball is red. Note that the argument doesn't apply to the red balls, which can change arbitrarily. Give 5 or 6 points if there is a minor omission or unclarity. Give 1 to 3 points if they made a useful start on a general result, or if they did several special cases and guessed a useful rule or otherwise seemed to be getting somewhere toward the answer.

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Solution for Problem 3, Michigan Math Prize Competition, December 1986

3. Let a, b, and c be 3 consecutive positive integers.
(2 points) (a) Show that ab cannot be the square of an integer.
(2 points) (b) Show that ac cannot be the square of an integer.
(6 points) (a) Show that abc cannot be the square of an integer.

(a) $a^2 < ab = a(a+1) < (a+1)^2$ so ab cannot be a perfect square.

(b) $a^2 < ac = a(a+2) < (a+1)^2$ so ac cannot be a perfect square.

(c) Notice that $\gcd(a,b) = \gcd(b,c) = 1$ and $\gcd(a,c) = 1$ or 2.
If abc is a perfect square, then its prime factorization has only even powers:

$$abc = 2^{2e_1} p_2^{2e_2} p_3^{2e_3} \ldots p_k^{2e_k} .$$

Each odd p_i divides only one of a, b, or c, and so $p_i^{2e_i}$ must divide that same factor.

Now if b is even, 2^{2e_1} must divide b and so a and b are both perfect squares contradicting part (a).

On the other hand, if b is odd, then a and c are both even. Now $\frac{a}{2}$ and $\frac{c}{2}$ are consecutive integers. Furthermore $\frac{a}{2}$, b, and $\frac{c}{2}$ are pairwise relatively prime, and their product equals $abc/4$ is another perfect square. Thus each of the three factors must be a perfect square. In particular, we must have $\frac{a}{2}$ and $\frac{c}{2}$ as consecutive perfect squares again violating part (a).

SUGGESTED SOLUTION TO PROBLEM #4 (INGRAHAM VIA BABCOCK)

Solution to Part a.): $x^2 + y^2 = 5$ is the circle at (0,0) with radius $\sqrt{5}$. $\sqrt{x} + \sqrt{y} = 2$ has graph lying entirely in the closed first quadrant, is continuous, and has (0,4) and (1,1) on its graph. Since the first of these points is outside the circle and the second inside, the Intermediate Value Theorem implies the graphs intersect. By symmetry, if (a,b) is a point of intersection, so is (b,a). Since no point of intersection of these two curves lies on $x = y$, we have at least two solutions.

Solution to Part b.): Because of the symmetry of solutions, the line joining the two points of intersection has slope -1. Therefore it is of the form $x + y = c$, and it only remains to determine c.

$\sqrt{x} + \sqrt{y} = 2$ implies $x + 2\sqrt{xy} + y = 4$ implies $4xy = [4-(x+y)]^2$.

Therefore $5 = x^2 + y^2 = (x+y)^2 - 2xy = (x+y)^2 - \frac{1}{2}[4 - (x+y)]^2$.

Substituting c for $x + y$ gives

$$5 = c^2 - \tfrac{1}{2}[4 - c]^2 = c^2 - 8 + 4c - \tfrac{1}{2}c^2 = \tfrac{1}{2}c^2 - 8 + 4c.$$

Hence we get

$$c = -4 \pm \sqrt{42}.$$

We see that c must be positive, so $x + y = -4 + \sqrt{42}$ is the solution.

MMPC - Part II

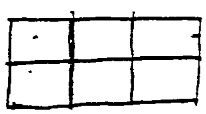
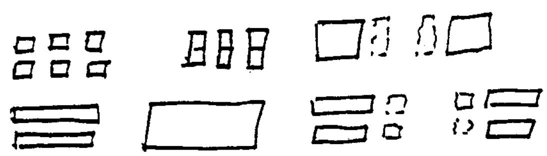
5. An $n \times m$ rectangle is tiled with unit squares. Let $R(n, m)$ denote the number of rectangles formed by the edges of these unit squares. Thus, $R(2, 1) = 3$.

2 (~~1~~ point) A) Find $R(2, 3)$

3 (~~2~~ points) B) Find $R(n, 1)$ for n a positive integer.

5 (~~3~~ points) C) Find a formula for $R(n, m)$. (with justification)

~~(4 points) D) Verify your answer to Part C.~~

A)  $R(2,3) = 18$ 

B) $R(n,1) = \binom{n+1}{2}$ since each pair of lines $\perp$ to the long side determine a unique rectangle

C) $R(n,m) = \binom{n+1}{2} \cdot \binom{m+1}{2}$

D). Each rectangle is uniquely determined by choosing 2 of the $(n+1)$ vertical lines and choosing 2 of the $(m+1)$ horizontal lines. Thus

$$R(n,m) = \binom{n+1}{2} \cdot \binom{m+1}{2}.$$