MMPC CODE NUMBER

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THIRTIETH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by the

MICHIGAN SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA

PART II

December 10, 1986

INSTRUCTIONS
(to be read aloud to participants by supervisor or proctor)

1. Carefully record your six digit MMPC code number in the upper righthand corner of this page. This is the only way to identify you with this test booklet. PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.

2. Part II consists of problems and proofs. You will be allowed 100 minutes for the five questions. To receive full credit for a problem, you are expected to justify your answer.

3. You are not expected to solve all problems completely. Look over all the problems and work first on those that interest you the most.

4. Each problem is on a separate page. You should show most of your work on that page. If it is necessary to use additional paper for your answer, indicate this on the exam page and write your identification number and the problem number in the upper righthand corner of each additional sheet.

5. If you are unable to completely solve a particular problem, partial credit may be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.

6. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate ideas of how to attack the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

7. The competition rules prohibit your asking questions of anyone during the examination. The use of notes, reference material, computational aids, or any other aid is likewise prohibited. When the supervisor announces that the 100 minutes have elapsed, please cease work immediately and insert all significant extra paper into the booklet. It is not necessary to return scratch paper on which routine numerical calculations have been made.

8. You may now open the test booklet and begin.

Score						
	1	2	3	4	5	TOTAL

1. $\triangle DEF$ is constructed from equilateral $\triangle ABC$ by choosing D on AB, E on BC, and F on CA so that

$$\frac{DB}{AB} = \frac{EC}{BC} = \frac{FA}{CA} = \alpha,$$ where α is a number between 0 and 1/2.

(3 points) (a) Show that $\triangle DEF$ is also equilateral.

(7 points) (b) Determine the value of α that makes the area of $\triangle DEF$ equal to one half the area of $\triangle ABC$.

2. A bowl contains some red balls and some white balls. The following operation is repeated until only one ball remains in the bowl:

> Two balls are drawn at random from the bowl. If they have different colors, then the red one is discarded and the white one is returned to the bowl. If they have the same color, then both are discarded and a red ball (from an outside supply of red balls) is added to the bowl.
>
> (Note that this operation--in either case--reduces the number of balls in the bowl by one.)

(3 points) (a) Show that if the bowl originally contained exactly 1 red ball and 2 white balls, then the color of the ball remaining at the end (i.e., after two applications of the operation) does not depend on chance, and determine the color of this remaining ball.

(7 points) (b) Suppose the bowl originally contained exactly 1986 red balls and 1986 white balls. Show again that the color of the ball remaining at the end does not depend on chance and determine its color.

3. Let a, b, and c be three consecutive positive integers, with $a < b < c$.

(2 points) (a) Show that ab cannot be the square of an integer.

(2 points) (b) Show that ac cannot be the square of an integer.

(6 points) (c) Show that abc cannot be the square of an integer.

4. Consider the system of equations

$$\sqrt{x} + \sqrt{y} = 2$$
$$x^2 + y^2 = 5$$

(3 points) (a) Show (algebraically or graphically) that there are two or more solutions in real numbers x and y.

(7 points) (b) The graphs of the two given equations intersect in exactly two points. Find the equation of the straight line passing through these two points of intersection.

5. Let n and m be positive integers. An n x m rectangle is tiled with unit squares. Let r(n, m) denote the number of rectangles formed by the edges of these unit squares. Thus, for example, r(2, 1) = 3.

(2 points) (a) Find r(2, 3).

(3 points) (b) Find r(n, 1).

(5 points) (c) Find, with justification, a formula for r(n, m).

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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ACKNOWLEDGEMENTS

The following corporations and professional organizations have contributed generously to this competition.

Arvco Container Corporation
Burroughs Corporation
Ford Motor Company
Kuhlman Corporation
Michigan Bell Telephone
Michigan Council of Teachers of Mathematics
The Upjohn Company

The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.