THIRTIETH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

PART I

October 8, 1986

INSTRUCTIONS

(to be read aloud to the students by the supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to insure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely and darkly.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "e) none of these" or "e) none of the above." If you believe none of the first four alternatives to be correct, mark e) in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request anyone to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes, or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You may now open the test booklet and begin.

1986 MMPC EXAM I

1. A certain version of BASIC allows real variable names subject to the three rules: (1) names must begin with an upper case letter, (2) names can have one or two characters, and (3) each character must be a numeral or an upper case letter. How many different real variable names does this version of BASIC allow?

   (a) 936
   (b) 33,696
   (c) 962
   (d) 1,247,714
   (e) 1296

2. The number of elements in the set $\{p \mid p \text{ is prime and } 100 \leq p \leq 110\}$ is

   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4

3. The number of elements in the set $\{n \mid 1 \leq n \leq 20 \text{ and } \sqrt{n} \text{ is irrational}\}$ is

   (a) 5
   (b) 8
   (c) 12
   (d) 14
   (e) 16

4. If

   $$(x - 1)(x - 2)(x - 3)(x - 4)(x - 5) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

   where a_0, a_1, a_2, a_3, and a_4 are constants, then a_4 =

   (a) 15
   (b) 5
   (c) -5
   (d) -15
   (e) none of these

5. Which of the following lines has no points in the Second Quadrant?

   (a) $y = 4x + 9$
   (b) $y = -4x + 9$
   (c) $y = 4x - 9$
   (d) $y = -4x - 9$
   (e) none of these

6. The set of all points in the (x,y)-plane that simultaneously satisfy the inequalities

$$x + 1 \geq 6,\ y - x \geq 0,\ y \geq 1,\ \text{and}\ y + x \leq 12$$

is

(a) the empty set
(b) a single point
(c) a triangle and its interior
(d) a parallelogram and its interior
(e) an infinite region bounded by three straight lines

7. What are the last two digits of 2^{100}?

(a) 00
(b) 16
(c) 24
(d) 48
(e) 76

8. Suppose the sum of two numbers is 16 and their difference is 3. Then the smaller of the two numbers is

(a) 6 2/3
(b) 5 1/2
(c) 5
(d) -4
(e) none of these

9. The number of <u>distinct</u> <u>real</u> numbers x that satisfy the equation

$$x^4 + 4x^3 + 6x^2 + 4x + 1 = 0 \text{ is}$$

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

10. For which of the following is $n^2 + n + 41$ <u>not</u> a prime number?

(a) n = 1
(b) n = 5
(c) n = 21
(d) n = 41
(e) none of these

11. If $a \geq 0$ and $b \geq 0$, then $\sqrt{a^2 b} \cdot \sqrt[3]{a^2 b^4}$ is equal to

(a) $a^{5/3} \; b^{4/3}$

(b) $a^{5/3} \; b^{7/3}$

(c) $a^{11} \; b^{11/3}$

(d) $\sqrt{a^4 \; b^5}$

(e) none of these

12. The function f is defined on the real numbers by

$f(x) = x^2 - 9$ if $x > 0$, $f(x) = x - 6$ if $-4 \leq x \leq 0$ and

$f(x) = 2x + 16$ if $x < -4$.

Then $f(f(f(1)))$ is equal to

(a) 4
(b) 0
(c) -6
(d) -7
(e) -8

13. The equation $x^2 + 6x + 25 = 0$ has solutions (where $i = \sqrt{-1}$)

(a) $3 \pm 4i$
(b) $4 \pm 3i$
(c) $-3 \pm 4i$
(d) $-4 \pm 3i$
(e) none of these

14. The graph of $y^2 = 3x^2 - 1$ is

(a) a parabola
(b) a circle
(c) an ellipse
(d) a hyperbola
(e) two intersecting straight lines

15. In a triangle with vertices A, B and C, the angle at A is 40° and the angle at B is 80°. What is the relationship between x = length of side BC and y = length of side AC?

(a) $y < x$
(b) $y = x$
(c) $x < y < 2x$
(d) $y = 2x$
(e) $y > 2x$

16. The area of the triangle with vertices (1,6), (1,11), and (7,43) is

(a) 15
(b) 30
(c) 51
(d) 90
(e) 129

17. An expression equal to $\frac{\sqrt{2}}{2 - \sqrt{2}}$ is

(a) 1/2

(b) 1

(c) $1 + \sqrt{2}$

(d) $2\sqrt{2} - 2$

(e) $\frac{2 - \sqrt{2}}{2}$

18. How many strings of ten letters consist of exactly seven A's and three B's (in some order)?

(a) 8
(b) 120
(c) 210
(d) 720
(e) 1024

19. Let L be the graph of $2x + y = 3$, and let M be the graph of $x - 2y = 4$. Which of the following statements is true about the lines L and M?

(a) They are parallel.
(b) They intersect at right angles.
(c) They intersect in the first quadrant.
(d) Neither enters the second quadrant.
(e) Both enter the second quadrant.

20. A student has an average (mean) score of 73 on 5 tests this semester. It is later discovered that one of the tests was misgraded, and rather than being an 81 it should have been a 91. What is the correct average of the 5 tests?

(a) 74
(b) 75
(c) 76
(d) 78
(e) 83

21. Which of the following is a factor of $2^{120} + 1$?

(a) $2 + 1$

(b) $2^{30} + 1$

(c) $2^{40} + 1$

(d) $2^{60} + 1$

(e) $2^{119} + 1$

22. A rectangle is measured and found to have length 14" and width 6", with both measurements reported to the nearest inch. The true area of the rectangle is therefore approximately 84 square inches. What is the maximum possible error in this approximation of the area?

(a) 1/4 sq. in.
(b) 1/2 sq. in.
(c) 1 sq. in.
(d) 2 sq. in.
(e) 10 1/4 sq. in.

23. As θ increases from 0 to π, the value of $\cos(\theta/2)$

(a) increases from 0 to 1
(b) decreases from 1 to 0
(c) decreases from 1 to -1
(d) increases from 0 to $\sqrt{2}/2$
(e) decreases from 1/2 to 0

24. The three angle bisectors of a triangle

(a) may possibly not have a common intersection
(b) must have a common intersection, but possibly at a point outside the triangle
(c) must have a common intersection at a point inside the triangle equidistant from the three sides
(d) must have a common intersection at a point inside the triangle equidistant from the three vertices
(e) must have a common intersection at a point inside the triangle 2/3 of the way along each bisector from vertex to opposite side

25. If $a_1 = a_2 = 1$ and $a_{n+2} = a_n/(1 + a_{n+1})$ for all $n > 0$, then $a_5 =$

(a) 5
(b) 1
(c) 5/6
(d) 2/3
(e) 3/10

26. Consider the function f defined on the set $\{x \mid x > 1\}$ by

$$f(x) = x^{\frac{1}{\ln x}}$$

where ln x is the natural (base e) logarithm of x. Then as x increases

(a) f(x) increases
(b) f(x) decreases
(c) f(x) remains constant
(d) f(x) increases for $x < e$ and decreases for $x > e$
(e) f(x) decreases for $x < e$ and increases for $x > e$

27. A man walked due west 800 feet, then due north 900 feet, then due east 1200 feet, and finally due south 700 feet. How far is he from his starting point?

(a) $\sqrt{200000}$ feet

(b) 600 feet

(c) $\sqrt{560000}$ feet

(d) 200 feet

(e) 3600 feet

28. If $16^x = (1/2)^{4-x}$ then x equals

(a) -4/3
(b) -4/5
(c) 4/3
(d) -1
(e) none of these

29. A child has 125 perfect cubes to use as building blocks. She stacks them to form a large block of size 5 x 5 x 5. From a certain point outside the pile it is possible to see three faces of the large block simultaneously. How many of the little cubes can be seen from this point?

(a) 36
(b) 61
(c) 64
(d) 89
(e) 100

30. $\sin(2 \text{ arcsec } 3) =$

(a) $\frac{4\sqrt{2}}{3}$

(b) $\frac{\sqrt{2}}{2}$

(c) $\frac{4\sqrt{2}}{9}$

(d) $\frac{2\sqrt{2}}{9}$

(e) cannot be determined without a calculator

31. Of five thin straight sticks of lengths 6 feet, 6.5 feet, $\sqrt{42}$ feet, 6.9 feet, and 7.5 feet, what is the length of the longest one that you could hide in a rectangular chest which is 2 feet high by 3 feet wide by 6 feet long?

(a) 6 feet

(b) 6.5 feet

(c) $\sqrt{42}$ feet

(d) 6.9 feet

(e) 7.5 feet

32. Suppose $\log_b a = c$, where $a > 0$, $b > 1$. Then $\log_b a^3 =$

(a) c^3

(b) $3c$

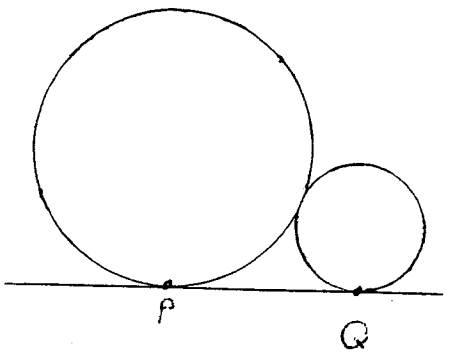
(c) $3 + c$

(d) a^3

(e) none of these

33. Circles of radius 4 and 9 are tangent to each other and to a horizontal line. Find the distance d(PQ) between the two points of tangency with the line.

(a) 6
(b) $\sqrt{56}$
(c) 12
(d) $9\sqrt{2}$
(e) 13

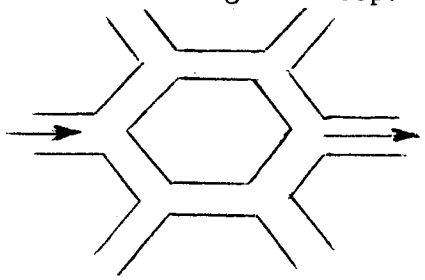


34. The proposition "$f(x) \neq 5$ unless x is prime" is logically equivalent to

(a) "if $f(x) = 5$, then x is prime"
(b) "if $f(x) \neq 5$, then x is prime"
(c) "if x is prime, then $f(x) = 5$"
(d) "if x is prime, then $f(x) \neq 5$"
(e) "if x is not prime, then $f(x) = 5$"

35. A maze for small lab animals is made in the form of a hexagonal loop with a side passage at each vertex. The start and finish are side passages at opposite vertices. A rat is put into the starting passage and starts running. It is equally likely to choose left or right openings at each vertex independently of all previous choices. What is the probability that it will go directly to the finish passage without entering any side passages, or going unnecessarily far around the hexagonal loop?

(a) 1/2
(b) 1/4
(c) 1/8
(d) 1/16
(e) 3/32



36. A company gave an executive a 10% raise for 1984. The executive received another 10% raise for 1985. For 1986 the salary was cut by 19%. Comparing the final (1986) salary with the pre-1984 salary, we find to the nearest 1% it is

(a) about 1% higher
(b) about 2% higher
(c) about 1% lower
(d) about 2% lower
(e) none of the above

37. What is the solution set of $|x| < x + 1$, where $|x|$ indicates the absolute value of the real number x?

(a) $\{ x \mid x > 0 \}$

(b) $\{ x \mid x < 0 \}$

(c) $\{ x \mid x > -1/2 \}$

(d) $\{ x \mid x < -1/2 \}$

(e) the set of all real numbers

38. Two equally matched baseball teams play a "best four out of seven" series. Assuming each team has a 50% chance of winning each game and that the outcomes of the games are independent of outcomes of earlier games, what is the probability that all seven games will need to be played to determine the champion?

(a) 63/64
(b) 6/7
(c) 1/2
(d) 3/7
(e) 5/16

39. The solution of $\frac{x + 2}{x - 1} < 2$ is

(a) $\{ x \mid x > 1 \}$

(b) $\{ x \mid x > 4 \}$

(c) $\{ x \mid x < 1 \}$

(d) $\{ x \mid 1 < x < 4 \}$

(e) $\{ x \mid x < 1 \text{ or } x > 4 \}$

40. The graph of the equation $(2x + 3y)^2 - (x + y)^2 = 1 + 10xy$ is

(a) a circle
(b) an ellipse
(c) a hyperbola
(d) a parabola
(e) two intersecting lines

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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