1. Old parks sometimes have a tetrahedral pile of cannon balls. That is, each layer is a tightly packed triangular layer of balls.

(3 points) A. How many cannon balls are in a tetrahedral pile of cannon balls of N layers?

(7 points) B. How high is a tetrahedral pile of cannon balls of N layers? (Assume each cannon ball is a sphere of radius R.)

A. The top layer has 1 ball.

The next to top layer has $1+2=3$ balls.

⋮

The N-the layer has $1+2+\cdots+N$ balls. That is, $\frac{N(N+1)}{2}$ balls.

Adding, the number of balls in N layers is

$$\sum_{i=1}^{N} \frac{i(i+1)}{2} = \frac{1}{2}\sum_{i=1}^{N} i^2 + \frac{1}{2}\sum_{i=1}^{n} i = \frac{1}{2}\left(\frac{N(N+1)(2N+1)}{6} + \frac{N(N+1)}{2}\right)$$

$$= \frac{1}{4}N(N+1)\left\{\frac{2N+1}{3}+1\right\} = \frac{1}{6}N(N+1)(N+2)$$

Full Credit

2 points

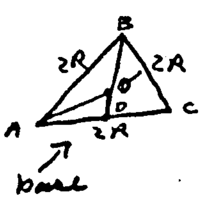
B. The centers of any four touching spheres form a regular tetrahedron. Let h be the heigth of this regular tetrahedron. Then the heigth of N laye[rs]

is $R + (N-1)h + R$

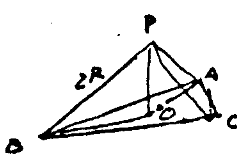
$= 2R + (N-1)h$

Suggestion: 5 points for this much

To find h we twice apply the Pythagorean theorem.



$BD = \sqrt{3}R$, so $BO = \frac{2}{3}\sqrt{3}R$



$PO = \text{altitude} = \sqrt{(2R)^2 - (\frac{2}{3}\sqrt{3}R)^2}$

$= 2\sqrt{\frac{2}{3}}R$

We can also find h via

$\sin 60° = R/x \Rightarrow x = \frac{2R}{\sqrt{3}}$

Then (in 3 space)

$2R = \sqrt{\frac{4R^2}{3} + h^2}$

⋮

So ans $= 2R + (N-1)\frac{2\sqrt{2}}{\sqrt{3}}R$

Full Credit

2. A prime is an integer greater than one whose only positive integer divisors are itself and one.

(3 points) A. Find a triple of primes (p, q, r) such that $p = q + 2$ and $q = r + 2$.

(7 points) B. Prove that there is only one triple (p, q, r) of primes such that $p = q + 2$ and $q = r + 2$.

A. Let $(p, q, r) = (7, 5, 3)$.

B. S'pose p, q, r are primes and $p = q+2$, $q = r+2$. Then

$(p, q, r) = (r+4, r+2, r)$.

Modulo 3, these numbers are congruent to

$$(r+1, r+2, r).$$

Now any three consecutive ~~i~~integers must have one of them divisible by 3. So 3 divides one of $r+1$, $r+2$ or r.

Thus 3 ~~Since r is prime we have 3~~ divides one of p, q, r.

But p, q, r are primes. So one of p, q, r must be ~~[illegible]~~ 3.

$p = 3 \Rightarrow r = -1$ which is not prime.

$q = 3 \Rightarrow r = 1$ which is not prime.

So $r = 3$, whence $(p, q, r) = (7, 5, 3)$.

3. The function g is defined recursively on the positive integers by $g(1) = 1$, and for $n > 1$, $g(n) = 1 + g(n - g(n - 1))$.

(2 points) A. Find $g(1)$, $g(2)$, $g(3)$ and $g(4)$.

1, 2, 2, 3

(3 points) B. Describe the pattern formed by the entire sequence $g(1)$, $g(2)$, $g(3)$, ...

(5 points) C. Prove your answer to Part B.

A. $g(1)=1$, $g(2)=1+g(1-g(1)) = 1+g(1) = 2$. $g(3)=1+g(3-g(2)) = 1+g(1) = 2$ and $g(4)=1+g(4-g(3)) = 1+g(2) = 3$.

B. $g(1), g(2), g(3), \ldots$ forms the pattern of one 1, followed by two 2s, followed by three 3s, ..., followed by n-repetitions of n, and so on.

C. Let $T_n = (1+2+\cdots+n) = \frac{n(n+1)}{2}$. Formally Part B says that $g(1)=1$ and $g(n)=k$ if $T_{k-1} < n \le T_k$. This we prove by induction on k, the value of $g(n)$.

If $n=1$, then $T_0 = 0 < 1 \le T_1 = 1$ and $g(1)=1$.

Note the induction hypothesis is

(*) $k \ge 1$ and $T_{k-1} < n \le T_k \Rightarrow g(n) = k$.

Suppose then that $T_k < m \le T_{k+1}$. We must show $g(m) = k+1$.

Note $g(T_k+1) = 1+g(T_k - g(T_k+1-1)) = 1+g(T_k - g(T_k))$

$= 1+g\left(\frac{k(k+1)}{2} - k\right)$ by (*).

$= 1+g\left(\frac{k^2-k}{2}\right) = 1+g(T_{k-1}) = 1+k$ by (*).

If $T_k+1 < m \le T_{k+1}$, then $g(m) \overset{1+}{=} g(m-g(m-1)) = 1+g(m-k)$

But $T_k < m \le T_{k+1} \Rightarrow T_k - k < m-k \le T_{k+1}-k$ or $T_{k-1} < m-k \le T_k+1$.

So $T_k < m \le T_{k+1} \Rightarrow g(m) = 1+g(m-k) = 1+k$.

Hence, by induction the assertion holds.

4. Let x, y, and z be <u>real</u> numbers such that

$$x + y + z = 1 \quad \text{and} \quad xyz = 3 .$$

(2 points) A. Prove that none of x, y, nor z can equal 1.

(8 points) B. Determine all values of x that can occur in solutions (where all three of x, y, and z are real numbers.)

A. Suppose $x = 1$. Then $y + z = 0$, so $xyz = -y^2 = 3$ which is impossible in the real numbers. Thus $x \neq 1$. A similar argument shows $y = 1$ and $z = 1$ are also impossible.

B. Suppose $x + y + z = 1$ and $xyz = 3$ with x, y, z real numbers. Then $x \neq 0, y \neq 0, z \neq 0$ and $z = \frac{3}{xy}$. Thus $x + y + \frac{3}{xy} = 1$ whence

$x^2 y + x y^2 + 3 = xy$ or $x \cdot y^2 + (x^2 - x) y + 3 = 0$. We solve this for y in terms of x.

Thus $y = \dfrac{-(x^2 - x) \pm \sqrt{(x^2-x)^2 - 12x}}{2x}$. We need $(x^2 - x)^2 - 12x \geq 0$.

Thus we need $x^4 - 2x^3 + x^2 - 12x \geq 0$. 4 Points

$x(x^3 - 2x^2 + x - 12) \geq 0$.

3 is a root of $x^3 - 2x^2 + x - 12$.

$$\begin{array}{r|l} & x^2 + x + 4 \\ \hline x - 3 & x^3 - 2x^2 + x - 12 \\ & \underline{x^3 - 3x^2} \\ & \quad x^2 + x \\ & \quad \underline{x^2 - 3x} \\ & \qquad 4x - 12 \end{array}$$

So we need

$x(x-3)(x^2 + x + 4) \geq 0$.

Now $x^2 + x + 4 > 0$ for all real x. So we need $x \neq 0$, $x(x-3) \geq 0$.

So the solution set is $(-\infty, 0) \cup [3, \infty)$. Full Credit

Suggestion: Give 2 points extra credit for verification that all $x \in (-\infty, 0) \cup [3, \infty)$ are solutions.

5. A round robin tournament was played among thirteen teams. Each team played every other team exactly once. At the conclusion of the tournament. it happened that each team had won six games and lost six games.

(2 points) A. How many games were played in this tournament?

(3 points) B. Define a *circular triangle* in a round robin tournament to be a set of three different teams in which none of the three teams beat both of the other two teams. How many *circular triangles* are there in this tournament?

(5 points) C. Prove your answer to Part B.

A. $\binom{13}{2} = \frac{13\cdot 12}{1\cdot 2} = 78$ (or $12+11+\cdots+2+1 = \frac{13\cdot 12}{2} = 78$)

B. One method is to first do problem C. Another method is to construct such a tourna & count triangles. EG. Number the teams $T0, T1, \ldots, T12$ and suppose Team Ti beats teams $Ti+1, \ldots, Ti+6$ where the "addition" is modulo 13. Then each team is part of $1+2+3+4+5+6$ circular triangles. So there are $\frac{1}{3}(13)(21) = 91$ circular triangl

C. Two solutions are given here:

(Grossman).

There are $\binom{13}{3}$ triangles (circular or not).

Of these, each team (say A) is part of a non-circular triangle (as in the figure)

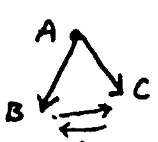


figure
A beats both of B,C.

either direction

in $\binom{6}{2} = 15$ ways. So the number of circular Δ's is $\binom{13}{3} - 13\cdot\binom{6}{2}$

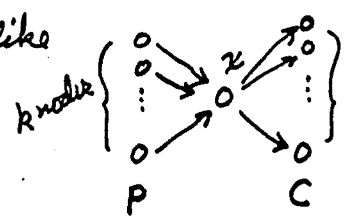
$= 286 - 195 = 91.$

Note this easily generalizes to $2k+1$ teams.

(Second Solution) Let the number of teams be 2

For A,B subsets of the nodes of the graph, let $\text{Arcs}_{A\to B}$ = the number of arcs from Set A to S

For each node x, the graph (locally) looks like



Full graph $= G = P \cup \{x\} \cup ($

P = teams defeating Team x, C = Teams beat by

Let $T(x) = \#$ circular triangles containing x.

Then $\text{Arcs}_{C\to P} = T(x)$.

Note $\text{Arcs}_{C\to G} = \text{Arcs}_{C\to C} + \text{Arcs}_{C\to P}$

$= \binom{k}{2} + T(x) = \text{Arcs}_{C\to C\cup P} =$

Solving, $T(x) = (k+1)k/2$. Since each tr is counted 3 times, the answer is $\frac{1}{3}(2k+1)(k$

$= \Sigma\, T(x)$, $x \in G$.