1. Old parks sometimes have a tetrahedral pile of cannon balls. That is, each layer is a tightly packed triangular layer of balls.

(3 points) A. How many cannon balls are in a tetrahedral pile of cannon balls of \( N \) layers?

(7 points) B. How high is a tetrahedral pile of cannon balls of \( N \) layers? (Assume each cannon ball is a sphere of radius \( R \).)

A. The top layer has 1 ball.

The next to top layer has \( 1+2=3 \) balls.

The \( N \)-the layer has \( 1+2+\cdots+N \) balls. That is, \( \frac{N(N+1)}{2} \) balls.

Adding, the number of balls in \( N \) layers is

\[
\sum_{i=1}^{N} \frac{i(i+1)}{2} = \frac{1}{2} \sum_{i=1}^{N} i^2 + \frac{1}{2} \sum_{i=1}^{N} i = \frac{1}{2} \left( \frac{N(N+1)(2N+1)}{6} + \frac{N(N+1)}{2} \right)
\]

\[
= \frac{1}{2} N(N+1) \left\{ \frac{2N+1}{3} + 1 \right\} = \frac{1}{6} N(N+1)(N+2)
\]

B. The centres of any four touching spheres form a regular tetrahedron. Let \( h \) be the height of this regular tetrahedron. Then the height of \( N \) layer is \( R + (N-1)h + R \)

\[
= 2R + (N-1)h
\]

To find \( h \), we twice apply the Pythagorean theorem.

\[
BD = \sqrt{3}R, \quad \text{so} \quad \theta \sigma = \frac{2}{3} \sqrt{3}R
\]

\[
PO = \text{altitude} = \sqrt{(2R)^2 - \left( \frac{2}{3} \sqrt{3}R \right)^2} = \frac{2\sqrt{3}}{3}R
\]

We can also find \( h \) via \( \sin \theta = \frac{R}{x} \Rightarrow x = \frac{2R}{\sqrt{3}} \)

Then (in 3 space)

\[
2R = \sqrt{h^2 + R^2}
\]

\[
\Rightarrow 2R = \sqrt{(\frac{2R}{\sqrt{3}})^2 + R^2}
\]

So and is \( 2R + (N-1) \frac{2\sqrt{2}}{\sqrt{3}} R \)
2. A prime is an integer greater than one whose only positive integer divisors are itself and one.

(3 points) A. Find a triple of primes \((p, q, r)\) such that \(p = q + 2\) and \(q = r + 2\).

(7 points) B. Prove that there is only one triple \((p, q, r)\) of primes such that \(p = q + 2\) and \(q = r + 2\).

A. Let \((p, q, r) = (7, 5, 3)\).

B. Suppose \(p, q, r\) are primes and \(p = q + 2, \ q = r + 2\). Then \((p, q, r) = (p+4, r+2, r)\).

Modulo 3, these numbers are congruent to \((r+1, r+2, r)\).

Now any three consecutive integers must have one of them divisible by 3. So 3 divides one of \(r+1, r+2\) or \(r\).

Thus 3 divides \(p, q, r\).

Since \(p, q, r\) are primes, 3 divides one of \(p, q, r\).

But \(p, q, r\) are primes. So one of \(p, q, r\) must be 3.

\[ p = 3 \Rightarrow r = -1 \quad \text{which is not prime} \]

\[ q = 3 \Rightarrow r = 1 \quad \text{which is not prime} \]

\[ s o r = 3, \quad \text{therefore} \quad (p, q, r) = (7, 5, 3) . \]
3. The function \( g \) is defined recursively on the positive integers by
\[ g(1) = 1, \quad g(n) = 1 + g(n - g(n - 1)) \]
for \( n > 1 \).

(2 points) A. Find \( g(1) \), \( g(2) \), \( g(3) \), and \( g(4) \).

\[
1, 2, 2, 3
\]

(3 points) B. Describe the pattern formed by the entire sequence \( g(1) \), \( g(2) \), \( g(3) \), ...

(5 points) C. Prove your answer to Part B.

A. \( g(1) = 1 \), 
\( g(2) = 1 + g(1 - g(1)) = 1 + g(1 - 1) = 1 + g(1) = 2 \).

\( g(3) = 1 + g(3 - g(2)) = 1 + g(1) = 2 \) and
\( g(4) = 1 + g(4 - g(3)) = 1 + g(2) = 3 \).

B. \( g(1), g(2), g(3), \ldots \) forms the pattern of
one 1, followed by two 2s, followed by three 3s, ...
followed by \( n \)-repetitions of \( n \), and so on.

C. Let \( T_n = (1 + 2 + \ldots + n) = \frac{n(n + 1)}{2} \). Formally, Part B says that
\( g(1) = 1 \) and \( g(n) = n \) if \( T_{n-1} < n \leq T_n \). This we prove by
induction on \( n \), the value of \( g(n) \).

If \( n = 1 \), then \( T_0 = 0 < 1 \leq T_1 = 1 \) and \( g(1) = 1 \).

Note the induction hypothesis is
\[
k \geq 1 \text{ and } T_{k-1} < n \leq T_k \implies g(n) = k.
\]

Suppose then that \( T_k < m \leq T_{k+1} \). We must show \( g(m) = k+1 \).

Note \( g(T_k + 1) = 1 + g(T_k - g(T_k + 1 - 1)) = 1 + g(T_k - g(T_k)) \)
\[
= 1 + g\left(\frac{n(n+1)}{2} - k\right) \text{ by } (k).
\]
\[
= 1 + g\left(\frac{n^2 - n - k}{2}\right) = 1 + g(T_{k-1}) = 1 + k \text{ by } (k).
\]

If \( T_k + 1 < m \leq T_{k+1} \), then \( g(m) = 1 + g(m - g(m - 1)) = 1 + g(m - k) \).

But \( T_k < m \leq T_{k+1} \implies T_k - k < m - k \leq T_{k+1} - k \) or \( T_{k-1} < m - k \leq T_k + 1 \).

So \( T_k < m \leq T_{k+1} \implies g(m) = 1 + g(m - k) = 1 + k \).

Hence, by induction, the assertion holds.
4. Let $x$, $y$, and $z$ be real numbers such that

$$x + y + z = 1 \text{ and } xyz = 3.$$ 

(2 points) A. Prove that none of $x$, $y$, nor $z$ can equal 1.

(8 points) B. Determine all values of $x$ that can occur in solutions (where all three of $x$, $y$, and $z$ are real numbers).

A. Suppose $x = 1$. Then $y + z = 0$, so $xyz = -y^2 = 3$
which is impossible in the real numbers. Thus $x \neq 1$. A similar argument shows $y = 1$ and $z = 1$ are also impossible.

B. Suppose $x + y + z = 1$ and $xyz = 3$ with $x, y, z$ real numbers.
Then $x \neq 0$, $y \neq 0$, $z \neq 0$ and $z = \frac{3}{xy}$. Thus $x + y + \frac{3}{xy} = 1$ whence

$$x^2y + xy^2 + 3 = xy$$
or $$xy^2 + (x^2-x)y + 3 = 0.$$ We solve this for $y$ in terms of $x$.
Thus $y = \frac{-x^2 + x \pm \sqrt{(x^2 + x)^2 - 12x}}{2x}$. We need $(x^2 - x)^2 - 12x \geq 0$.

Thus we need $x^4 - 2x^3 + x^2 - 12x \geq 0$.

$x(x^3 - 2x^2 + x - 12) \geq 0$.

3 is a root of $x^3 - 2x^2 + x - 12$.

So we need $x(x-3)(x^2 + x + 4) \geq 0$.

Now $x^2 + x + 4 > 0$ for all real $x$. So we need $x \neq 0$, $x(x-3) \geq 0$.

So the solution set is $(-\infty, 0) \cup [3, \infty)$. Full Credit

Suggestion: Give 2 points extra credit for verification that all $x \in (-\infty, 0) \cup [3, \infty)$ are solutions.
5. A round robin tournament was played among thirteen teams. Each team played every other team exactly once. At the conclusion of the tournament, it happened that each team had won six games and lost six games.

(2 points) A. How many games were played in this tournament?

(3 points) B. Define a circular triangle in a round robin tournament to be a set of three different teams in which none of the three teams beat both of the other two teams. How many circular triangles are there in this tournament?

(5 points) C. Prove your answer to Part B.

A. \( \binom{13}{2} = \frac{13 \times 12}{1 \times 2} = 78 \) (or \( 13 + 11 + \ldots + 2 + 1 = \frac{13 \times 12}{2} = 78 \))

B. One method is to first do problem C. Another method is to construct such a tournament & count triangles. E.g. Number the teams T0, T1, \ldots, T12 and suppose Team T1 beats teams T0, T1, \ldots, T6 where the “addition” is modulo 13. Then each team is part of 1+2+3+4+5+6 circular triangles, so there are \( \frac{1}{3} (13 \times 26) = 91 \) circular triangles.

C. Two solutions are given here:

(Grossman).

There are \( \binom{13}{3} \) triangles (circular or not).

Of these, each team (say A) is part of a non-circular triangle (as in the figure)

in \( \binom{6}{2} = 15 \) ways, so the number of circular A's is \( \binom{13}{3} - \binom{6}{2} \)

= 286 - 195 = 91.

Note: this easily generalizes to 2k+1 teams.

(Second Solution) Let the number of teams be 2k.

For A, B subsets of the nodes of the graph, let

\[ \text{Area}_{A+B} = \text{the number of arcs from set A to set B} \]

For each node \( x \), the graph (locally) looks like

\[ \text{Full graph} \]

\[ P \]

\[ C \]

\[ \text{P=teams defeating Team x} \]

\[ C=Teams beaten by x} \]

Let \( T(x) = \) # circular triangles containing x.

Then \( \text{Area}_{C \rightarrow P} = T(x) \).

Note \( \text{Area}_{C \rightarrow P} = \text{Area}_{C \rightarrow C} + \text{Area}_{C \rightarrow P} \)

\[ = \binom{k}{2} + T(x) = \text{Area}_{C \rightarrow C \rightarrow P} \]

Solving, \( T(x) = \binom{k}{2} - T(x) = \frac{k}{2} \). Since each triangle is counted 3 times, the answer is \( \frac{1}{3} \binom{2k+1}{k} \), \( k \in \mathbb{Z} \).