TWENTY-NINTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

PART I

October 9, 1985

INSTRUCTIONS

(to be read aloud to the students by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to insure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "e) none of these" or "e) none of the above." If you believe none of the first four alternatives to be correct, mark e in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request any one to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes, or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You may now open the test booklet and begin.

# 1985 MMPC EXAM I

1. The sum $1 + 2 + \cdots + 999 + 1000$ of the first one thousand positive integers is

   a) 500,500 b) 100,999 c) 512,215 d) 487,490 e) none of these

2. The largest integer $n$ such that $10^n$ evenly divides $25! = 25 \times 24 \times \cdots \times 2 \times 1$ is

   a) 5 b) 6 c) 7 d) 8 e) 9

3. Given $A = 1/3$, $B = \sqrt{0.01}$ and $C = 1/11$, which of the following is true?

   a) $B \le C \le A$
   b) $A \le B \le C$
   c) $C \le B \le A$
   d) $A > B$ and $C > B$
   e) none of the above

4. The number $\log_{10} 1000$ is equal to

   a) $-2$ b) 2 c) 3 d) 100 e) none of these

5. The inequality $3x + 1 < 4(x - 2)$ is equivalent to

   a) $4 < x$ b) $3 < x$ c) $x < 4$ d) $x < 9$ e) $9 < x$

6. The function $H$ is defined recursively on the positive integers by $H(1) = 1$ and for $n > 1$, $H(n) = n + H(n - 1)$. Then $H(H(2))$ is

   a) 2 b) 4 c) 6 d) 8 e) 10

7. If $\frac{5}{x} = \frac{4}{3}$, then $x$ is

   a) 4/5 b) 2 c) 3 d) 15/4 e) 4

8. The expression $\frac{1}{v} - \frac{4}{u}$ is equal to

   a) $\frac{-3}{v - u}$ b) $\frac{3}{v + u}$ c) $\frac{u - 4v}{uv}$ d) $\frac{-3}{uv}$ e) $\frac{v - 4u}{uv}$

9. For real numbers $x$, the largest value of $y = 5 - 2x - x^2$ is

a) 6

b) 5

c) 4

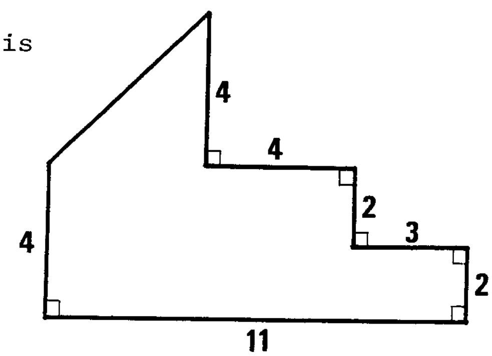
d) There is no largest value.

e) Not enough information is given to determine the largest value.

10. If $x$ and $y$ are non-negative real numbers, then the expression $\sqrt{50x^6y^8}$ is equal to

a) $25x^6y^8$  b) $5x^3y^4$  c) $25x^3y^4$  d) $5\sqrt{2}\,x^6y^8$  e) $5\sqrt{2}\,x^3y^4$

11. The area of the region in the figure is




a) 44

b) 46

c) 52

d) 54

e) not determined by the figure

12. The number of three element subsets of the set $\{1, 2, \cdots, 20\}$ of the first twenty positive integers is

a) 60  b) 210  c) 1140  d) 6840  e) 8000

13. Each of the following is the equation of a straight line. Of these, which line has the greatest slope?

a) $10x - y = 10$

b) $21x - 2y = 12$

c) $x + y = 14$

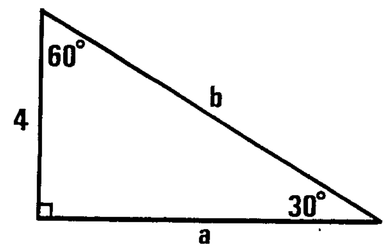
d) $x - 10y = 16$

e) $2x + 21y = 18$

14. The equation $8x^2 + 36x + 41 = 0$ has solutions: $(i = \sqrt{-1})$

a) $\frac{-9}{2} \pm \frac{i}{2}$  b) $\frac{9}{2} \pm \frac{i}{2}$  c) $\frac{-9}{4} \pm \frac{i}{4}$  d) $\frac{9}{4} \pm \frac{i}{4}$  e) none of these

15. Find the two lengths marked a and b in the figure.




a) $a = 4\sqrt{3}$ , $b = 8$

b) $a = 8/\sqrt{3}$ , $b = 8$

c) $a = 4$ , $b = 4\sqrt{2}$

d) $a = \sqrt{3}$ , $b = 2$

e) none of the above

16. The x-coordinate of the intersection of the two lines $2x + 3y = 1$ and $x - y = 3$ is $x =$

a) $-2$  b) $-1$  c) $1$  d) $2$  e) $3$

17. Find all values of $x$ for which the determinant $\begin{vmatrix} x & 3 \\ 2 & x \end{vmatrix}$ has value $3$ .

a) $x = 2$ and $x = 3$

b) $x = -2$ and $x = -3$

c) $x = -3$ and $x = 3$

d) $x = 2$ and $x = -2$

e) none of the above

18. For what values of $x$ does $\sqrt{-x} + \sqrt{x + 2}$ represent a real number?

a) no real $x$  b) all real $x$  c) $x \leq 0$  d) $x \geq -2$  e) $-2 \leq x \leq 0$

19. If $N$ is a positive integer that is exactly divisible by 120 and exactly divisible by 140, then the smallest $N$ could be is

a) between 100 and 200

b) between 200 and 400

c) between 400 and 1,000

d) between 1,000 and 2,000

e) between 2,000 and 20,000

20. If $A = \{1, 2, 3\}$ , $B = \{2, 4, 6\}$ and $C = \{1, 3, 5\}$ , then the set $(A \cup B) \cap C$ is

a) $\{2, 4, 6\}$  b) $\{1, 3\}$  c) $\{2, 5\}$  d) $\{1, 2, 4\}$  e) $\{1, 2, 3, 4, 5, 6\}$

21. How many angles $\theta$ between $0°$ and $360°$ satisfy the equation $|\cos(\theta)| = |\sin(\theta)|$ ?

a) 0 b) 1 c) 2 d) 4 e) infinitely many

22. A sphere of diameter 10 cm is placed inside a cube, 10 cm on each edge. What is the distance from a vertex (corner) of the cube to the nearest point on the sphere?

a) 5 cm b) $5(\sqrt{3} - 1)$cm c) $5(\sqrt{2} - 1)$cm d) $5\sqrt{3}$ cm e) $5\sqrt{2}$ cm

23. The polynomial $x^4 + 4x^3 - 11x^2 - 19x + 28$ has four distinct real roots, all of them irrational numbers. The sum of these four numbers is

a) $-4$ b) 19 c) 28 d) 0 e) an irrational number

24. As $x$ ranges over all real numbers, what are the possible values of

$y = \dfrac{x^2 + 2}{2x^2 + 1}$ ?

a) $\frac{1}{2} < y \leq 2$ b) $0 < y \leq 2$ c) $0 < y$ d) $\frac{1}{2} < y$ e) $y \leq 2$

25. The complex number $\dfrac{1 + i}{1 - i}$ is equal to

a) $1 - i$ b) 0 c) $-i$ d) $i$ e) $-1$

26. If it is not true that all fraptonians are gregorticular, then one can conclude that

a) All fraptonians are not gregorticular.
b) Some fraptonians are gregorticular.
c) Some fraptonians are not gregorticular.
d) Some gregorticular things are fraptonians.
e) All gregorticular things are fraptonians.

27. What number will be written when one follows the following algorithm?

Step 1: Start with $N = 0$ and $I = 6$ .
Step 2: If $I = 0$ then write the value of $N$ and stop.
Step 3: Replace $N$ with the value of $N + 1$ .
Step 4: Replace $I$ with the value of $I - N$ .
Step 5: Go to Step 2 and continue getting instructions.

a) 6 b) 3 c) 1 d) 0 e) none of these

28. The solution set of the inequality $1 < |x - 7| < 3$, when represented on the real number line, consists of

a) an interval of length 2

b) an interval of length 4

c) an interval of length 6

d) two distinct intervals, each of length 1

e) two distinct intervals, each of length 2

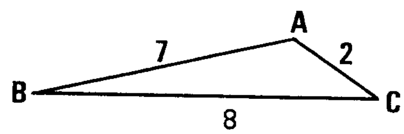
29. The ratio of the area of a circle to the area of an equilateral triangle inscribed in that circle is

a) $4\pi/(3\sqrt{3})$  b) $\pi$  c) $\pi/2$  d) $3\sqrt{3}/2$  e) $\pi/\sqrt{3}$

30. The sum of three distinct numbers is 100. The larger two numbers differ by 10. The smaller two numbers differ by 2. Then the largest of these three numbers is

a) 43 1/3  b) 40 2/3  c) 39 2/3  d) 35 1/3  e) none of these

31. In triangle ABC, cos(A) =




a) 7/8  b) -7/8  c) 11/28  d) -11/28  e) none of these

32. Define the operation # on pairs of real numbers by a # b = a + 2b + 3ab. How many real numbers satisfy the relation (a # a) # a = a # (a # a) ?

a) 0  b) 1  c) 2  d) infinitely many  e) none of these

33. How many four letter code words can be made from the 26 upper case English letters ('A' ,..., 'Z') in such a way that no two adjacent letters are the same?

a) $26 \cdot 25 \cdot 24 \cdot 23$  b) $26 \cdot 25^3$  c) $26^2 \cdot 25^2$  d) $25^4$  e) $26^4$

34. A 7 gram point mass is dropped from a height of 10 meters. After each bounce it rises a distance of one-third its previous height. The total distance it travels is

a) 20 m.  b) 30 m.  c) 33 1/3 m.  d) 66 2/3 m.  e) none of these

35. If $8^9 + 7^9 + 6^9$ is divided by 5, the remainder is

a) 0  b) 1  c) 2  d) 3  e) 4

36. A box contains five marbles: two are red, two are white, and one is blue. Two marbles are drawn at random from the box without replacing the first before drawing the second. What is the probability the two marbles have the same color?

a) 1/20 b) 1/10 c) 1/5 d) 1/4 e) 1/3

37. The three altitudes of a triangle must meet at a point

a) which bisects each altitude

b) 2/3 of the way along each altitude from vertex towards opposite side

c) equidistant from the three sides of the triangle

d) equidistant from the three vertices of the triangle

e) which may fall outside the triangle

38. Four plain 3 x 5 cards are prepared as follows. One of them has an 'X' marked on both sides. Each of the other three has an 'X' marked on only one side. The four cards are shuffled and (without looking) one of them is chosen at random and placed flat upon a table. If there is an 'X' marked on the visible side of the chosen card, what is the probability that there is an 'X' marked on its hidden side?

a) 1/4 b) 1/3 c) 1/2 d) 2/5 e) 3/8

39. A flagpole casts a shadow 6 feet high on a wall 10 feet away (see figure). At the same time, a 5 1/2 foot tall person casts a shadow 5 feet long on the flat ground. How high is the flagpole?

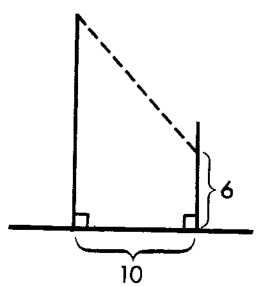
a) 11 feet

b) 11 1/2 feet

c) 15 1/2 feet

d) 16 feet

e) 17 feet




40. If ABCDEF is a 6-digit numeral that uses only the digits 0, 1, 2, 3, and 4, with $A \neq 0$ , then approximately what is the ratio of the number this numeral represents in Base 10 to the number this numeral represents in Base 5?

a) 2 b) 5 c) 6 d) 10 e) larger than 20

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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