28th Annual Michigan Mathematics Prize Competition

Part II Solutions

1. If \( m^2 = 2^6 + 2^9 + 2^n = 576 + 2^n = (24)^2 + 2^n \), then
\[ 2^n = m^2 - 24^2 = (m - 24)(m + 24) \text{, so that} \]
\[ 2^2 = m + 24, \quad 2^k = m - 24 \text{ where } 0 \leq k < \ell \quad \text{and } k + \ell = n. \]
Hence, \( 2^2 - 2^k = 48 \); but then \( 2^k \) divides \( 48 \Rightarrow k \leq 4 \).

Trying \( k = 0, 1, 2, 3, 4 \) gives \( k = 4, \ell = 6 \Rightarrow n = 10 \).

2. If \( N = 2M \), and \( x, y \) are any two integers, \( 1 \leq x, y \leq N \), then
\[ P(x \text{ is even}) = P(x \text{ is odd}) = P(y \text{ odd}) = P(y \text{ even}) = \frac{1}{2}. \]
Then
\[ P(x + y \text{ even}) = P(x \text{ odd}, y \text{ odd}) + P(x \text{ even}, y \text{ even}) \]
\[ = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \]

But, \( P(x + y \text{ odd}) = 1 - P(x + y \text{ even}) = \frac{1}{2} \)
\[ \therefore P(x + y \text{ even}) > P(x + y \text{ odd}). \]

If \( N = 2M + 1 \), then
\[ P(x \text{ even}) = \frac{M}{2M + 1} = P(y \text{ even}) \quad ; \quad P(x \text{ odd}) = \frac{M + 1}{2M + 1} = P(y \text{ odd}) \]

Thus, \( P(x + y \text{ even}) = P(x \text{ odd}, y \text{ odd}) + P(x \text{ even}, y \text{ even}) \)
\[ = \frac{M + 1}{2M + 1} \cdot \frac{M + 1}{2M + 1} + \frac{M}{2M + 1} \cdot \frac{M}{2M + 1} \]
\[ = \frac{(M + 1)^2 + M^2}{(2M + 1)^2} = \frac{2M^2 + 2M + 1}{4M^2 + 4M + 1} > \frac{1}{2} \]

Hence \( P(x + y \text{ odd}) = 1 - P(x + y \text{ even}) \)
\[ < \frac{1}{2} \quad \text{Hence: } P(\text{even}) > P(\text{odd}). \]

(or directly, \( P(x + y \text{ odd}) = \frac{2M(M + 1)}{(M + 1)^2} < \frac{1}{2} \)).
Let \( Y' \) be the reflection of \( Y \) on the \( X \)-axis and \( C \) the circle centered at \( Y' \) and tangent to the \( X \)-axis. Let \( XT \) be the "left-hand" tangent line from \( X \) to \( C \). Then \( M \) is the intersection of \( XT \) with the \( X \)-axis.

For \( \angle AMX = \angle BMT \)
\[
= \angle Y' MT + \angle Y'MB
\]
\[
= 2 \angle Y'MB, \text{ since } Y'N \text{ is the bisector of } \angle BMT
\]
\[
= 2 \angle YMB \text{ by reflection.}
\]

4. **Claim #1** \((a - b\sqrt{3})^n = a_n - b_n\sqrt{3} = (a - b\sqrt{3})^n\). easily shown by induction.

For reference, we need (*) \[ a_{n+1} = a_n + 3bb \]
\[ b_{n+1} = b_n + ab \]

**Claim #2**: \( a_n - b_n\sqrt{3} > 0 \) since \( a = a - b\sqrt{3} = a - b\sqrt{3} > 0 \) by hypothesis.

Hence \( \frac{a_n}{b_n} > \sqrt{3} \) for all \( n \).

**Claim #3**: \[ \frac{a_n}{b_n} = \frac{a_n + 1}{b_n + 1} \]; cross-multiplying, this is equivalent to \( a_{n+1} - b_{n+1} \geq 0 \). Using (*) we get
\[ a_n(b_n + ab_n) - b_n(aa + 3bb) \geq 0 \]
i.e. \( b(a_n + b\sqrt{3})(a_n - b\sqrt{3}) \geq 0 \), i.e. \( ba_n^a - a_n^b \geq 0 \) which is true.

Hence the sequence \( \frac{a_n}{b_n} \) is positive, decreasing and bounded below by \( \sqrt{3} \), and hence converges to \( L \).

Now \[ L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{a_{n+1}}{b_{n+1}} = \lim_{n \to \infty} \frac{a_n + 3b_n}{b_n + ab_n} \]

\[ = \lim_{n \to \infty} \frac{a_n + 3b}{b + ab_n} = \frac{aL + 3b}{bl + a} \]

Hence \[ bl^2 + aL = aL + 3b \]

\[ \Rightarrow bl^2 = 3b \]

\[ \Rightarrow l^2 = 3 \]

\[ \Rightarrow L = \pm \sqrt{3}. \text{ But } L \text{ is positive so } L = \sqrt{3}. \]

5. The obvious two triangles are \([ab, bn; mn]\) and \([cm, dm; mn]\), where we take \(a < b, c < d\). (Note that \(a = b, c = d\) is not possible.) But these triangles are not distinct if \(an = cm\) and \(bn = dm\) or, equivalently, if \(\frac{a}{b} = \frac{c}{d}\), i.e., the triangles are similar. For this case, we say

\[(mn)^2 = m^2n^2 = (a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2.\]

The condition of similarity gives \(ad - bc = 0\). But the other choice, \(x = ad + bc, y = bd - ac\) (remember that \(a < b, c < d\)) now yields a true triangle, which is the desired second solution.

**Example:** \(a = c = 3, b = d = 4, m = n = 5\)

First Triangle: \([15, 20; 25]\)

Second Triangle: \(x = 3.4 + 4.3, y = 4.4 - 3.3\) gives \([7, 24; 25]\)
Notice that we usually will get at least four triangles with hypotenuse \( mn \).
(try it with [3, 4; 5] and [5, 12; 13])

Need to show the second triangle is indeed distinct. For the similarity case, \( c = \lambda a, d = \lambda b, n = \lambda m \). Then \( an = cm = \lambda am; \ bn = dm = \lambda bm; \ mn = \lambda m^2 \).

But \( x = 2\lambda ab, y = \lambda b^2 - \lambda a^2 \).

If \( x = an \), then \( 2\lambda ab = \lambda am = m = 2b \Rightarrow (m^2 = b^2 + a^2) \Rightarrow 4b^2 = b^2 + a^2 \Rightarrow 3b^2 = a^2 \), impossible

If \( x = bm \), then \( 2\lambda ab = \lambda bm \Rightarrow m = 2a \Rightarrow (m^2 = b^2 + a^2) \Rightarrow 4a^2 = b^2 + a^2 \Rightarrow 3a^2 = b^2 \), impossible

Thus the triangle \([x, y; mn]\) is distinct.