28th Annual Michigan Mathematics
Prize Competition

Part II Solutions

1. If $m^2 = 2^6 + 2^9 + 2^n = 576 + 2^n = (24)^2 + 2^n$, then
$2^n = m^2 - 24^2 = (m - 24)(m + 24)$, so that
$2^\ell = m + 24$, $2^k = m - 24$ where $0 \le k < \ell$ and $k + \ell = n$.
Hence, $2^\ell - 2^k = 48$; but then 2^k divides $48 \Rightarrow k \le 4$.
Trying $k = 0, 1, 2, 3, 4$ gives $k = 4$, $\ell = 6$ $\therefore$ $\underline{n = 10}$.

2. If $N = 2M$, and x,y are any two integers, $1 \le x,y \le N$, then
$P(x \text{ is even}) = P(x \text{ is odd}) = P(y \text{ odd}) = P(y \text{ even}) = 1/2$. Then
$P(x + y \text{ even}) = P(x \text{ odd}, y \text{ odd}) + P(x \text{ even}, y \text{ even})$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

But, $P(x + y \text{ odd}) = 1 - P(x + y \text{ even}) = \frac{1}{2}$

$\therefore P(x + y \text{ even}) \ge P(x + y \text{ odd})$.

If $N = 2M + 1$, then

$P(x \text{ even}) = \dfrac{M}{2M + 1} = P(y \text{ even})$; $P(x \text{ odd}) = \dfrac{M + 1}{2M + 1} = P(y \text{ odd})$

Thus, $P(x + y \text{ even}) = P(x \text{ odd}, y \text{ odd}) + P(x \text{ even}, y \text{ evan})$

$$= \frac{M + 1}{2M + 1} \cdot \frac{M + 1}{2M + 1} + \frac{M}{2M + 1} \cdot \frac{M}{2M + 1}$$

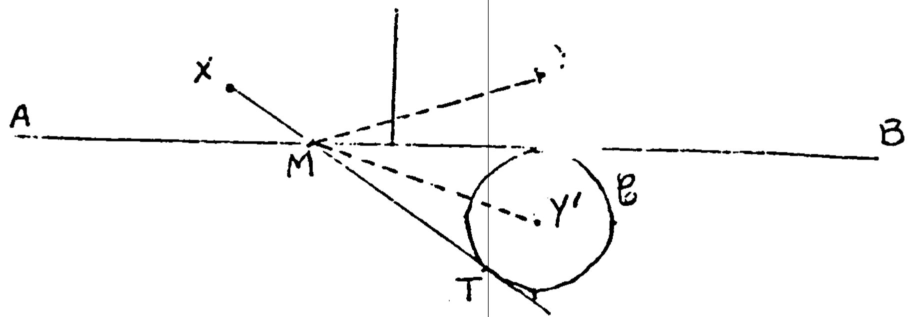
$$= \frac{(M + 1)^2 + M^2}{(2M + 1)^2} = \frac{2M^2 + 2M + 1}{4M^2 + 4M + 1} > \frac{1}{2}$$

Hence $P(x + y \text{ odd}) = 1 - P(x + y \text{ even})$

$< \frac{1}{2}$ Hence: $P(\text{even}) \ge P(\text{odd})$.

(or directly, $P(x + y \text{ odd}) = \dfrac{2M(M + 1)}{(M + 1)^2} < \dfrac{1}{2}$

over

3. 

Let Y' be the reflection of Y on the X-axis and C the circle centered at Y' and tangent to the X-axis. Let XT be the "left-hand" tangent line from X to C . Then M is the intersection of XT with the X-axis .

For $\angle AMX = \angle BMT$

$= \angle Y'MT + \angle Y'MB$

$= 2\angle Y'MB$, since Y'M is the bisector of $\angle BMT$

$= 2\angle YMB$ by reflection.

4. <u>Claim #1</u> $(a - b\sqrt{3})^n = a_n - b_n\sqrt{3} = (\overline{\alpha})^n$. easily shown by induction.

For reference, we need (*) $a_{n+1} = aa_n + 3bb_n$

$b_{n+1} = ba_n + ab_n$

<u>Claim #2</u>: $a_n - b_n\sqrt{3} \geq 0$ since $\overline{\alpha} = a - b\sqrt{3} \geq 0$ by hypothesis .

Hence $\dfrac{a_n}{b_n} \geq \sqrt{3}$ for all n .

<u>Claim #3</u>: $\dfrac{a_n}{b_n} = \dfrac{a_{n+1}}{b_{n+1}}$; cross-multiplying, this is equivalent to

$a_n b_{n+1} - b_n a_{n+1} \geq 0$. Using (*) we get

$a_n(ba_n + ab_n) - b_n(aa_n + 3bb_n) \geq 0$

i.e. $b(a_n + b_n\sqrt{3})(a_n - b_n\sqrt{3}) \geq 0$, i.e. $b\alpha^n\overline{\alpha}^n \geq 0$ which is true.

Hence the sequence $\dfrac{a_n}{b_n}$ is positive, decreasing and bounded below by $\sqrt{3}$, and hence converges to L

Now $L = \lim \frac{a_n}{b_n} = \lim \frac{a_{n+1}}{b_{n+1}} = \lim \frac{aa_n + 3bb_n}{ba_n + ab_n}$

$$= \lim \frac{a \cdot \frac{a_n}{b_n} + 3b}{b \cdot \frac{a_n}{b_n} + a} = \frac{aL + 3b}{bL + a}$$

Hence $bL^2 + aL = aL + 3b$

$\Rightarrow \quad bL^2 = 3b$

$\Rightarrow \quad L^2 = 3$

$\Rightarrow \quad L = \pm\sqrt{3}$. But L is positive so $L = \sqrt{3}$.

5. The obvious two triangles are $[ab, bn; mn]$ and $[cm, dm; mn]$, where we take $a < b$, $c < d$. (Note that $a = b$, $c = d$ is not possible.) But these triangles are not distinct if $an = cm$ and $bn = dm$ or, equivalently, if $\frac{a}{b} = \frac{c}{d}$, ie., the triangles are similar. For this case, we say

$$(mn)^2 = m^2n^2 = (a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2 .$$

The condition of similarity gives $ad - bc = 0$. But the other choice, $x = ad + bc$, $y = bd - ac$ (remember that $a < b$, $c < d$) now yields a true triangle, which is the desired second solution.

— <u>Example:</u> $a = c = 3$, $b = d = 4$, $m = n = 5$

First Triangle: $[15, 20; 25]$

Second Triangle: $x = 3.4 + 4.3$, $y = 4.4 - 3.3$ gives $[7, 24; 25]$

over

Notice that we usually will get at least four triangles with hypotenuse mn. (try it with [3, 4; 5] and [5, 12; 13])

Need to show the second triangle is indeed distinct. For the similarity case, $c = \lambda a$, $d = \lambda b$, $n = \lambda m$. Then $an = cm = \lambda am$; $bn = dm = \lambda bm$; $mn = \lambda m^2$. But $x = 2\lambda ab$, $y = \lambda b^2 - \lambda a^2$.

If $x = an$, then $2\lambda ab = \lambda am \Rightarrow m = 2b \Rightarrow (m^2 = b^2 + a^2)$

$$\Rightarrow 4b^2 = b^2 + a^2$$

$$\Rightarrow 3b^2 = a^2, \text{ impossible}$$

If $x = bm$, then $2\lambda ab = \lambda bm \Rightarrow m = 2a \Rightarrow (m^2 = b^2 + a^2)$

$$\Rightarrow 4a^2 = b^2 + a^2$$

$$\Rightarrow 3a^2 = b^2, \text{ impossible}$$

Thus the triangle $[x, y; mn]$ is distinct.