

TWENTY-EIGHTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by the

MATHEMATICAL ASSOCIATION OF AMERICA, MICHIGAN SECTION

PART II

December 5, 1984

INSTRUCTIONS

(to be read aloud to class by supervisor or proctor)

1. Carefully record your six digit MMPC code number in the upper righthand corner of this page. This is the only way to identify you with this test booklet. PLEASE DO NOT WRITE YOUR NAME ON THIS BOOKLET.
2. Part II consists of problems and proofs. You will be allowed 100 minutes for the five questions. To receive full credit for a problem, you are expected to justify your answer.
3. You are not expected to solve all problems completely. Look over all the problems and work first on those which interest you the most.
4. Each problem is on a different page. You should show most of your work on that page. If it is necessary to use additional paper for your answer, indicate this on the exam page and write your identification number and the problem in the upper righthand corner of each additional sheet.
5. If you are unable to completely solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved. If you have difficulty understanding what is required in a given problem, note this on your answer sheet and attempt to make a nontrivial restatement of the problem. Then try to solve the restated problem.
6. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate ideas of how to attack the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.
7. The competition rules do not allow anyone to answer any questions. The use of notes, reference material, computational aids, or any other aid is prohibited. When the supervisor announces that the 100 minutes are up, please cease work immediately and insert all significant extra paper into the booklet. It is not necessary to return scratch paper on which routine numerical calculations were made.
8. You may now open the test booklet and begin.

Score \_\_\_\_\_

1	2	3	4	5	TOTAL
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1. For what integers  $n$  is  $2^6 + 2^9 + 2^n$  the square of an integer?

- Two integers are chosen at random (independently, with repetition allowed) from the set  $\{1, 2, 3, \dots, N\}$ . Show that the probability that the sum of the two integers is even is not less than the probability that the sum is odd.

3. Let  $X$  be a point in the second quadrant of the plane and let  $Y$  be a point in the first quadrant. Locate the point  $M$  on the  $x$ -axis such that the angle  $\angle XM$  makes with the negative end of the  $x$ -axis is twice the angle  $\angle YM$  makes with the positive end of the  $x$ -axis.

4. Let  $a, b$  be positive integers such that  $a \geq b\sqrt{3}$ . Let  $\alpha^n = (a + b\sqrt{3})^n = a_n + b_n\sqrt{3}$  for  $n = 1, 2, 3, \dots$

i. Prove that  $\lim_{n \rightarrow +\infty} \frac{a_n}{b_n}$  exists.

ii. Evaluate this limit.

5. Suppose  $m$  and  $n$  are the hypotenuses of Pythagorean triangles, i.e., there are positive integers  $a, b, c, d$ , so that  $m^2 = a^2 + b^2$  and  $n^2 = c^2 + d^2$ . Show that  $mn$  is the hypotenuse of at least two distinct Pythagorean triangles.

Hint: you may not assume that the pair  $(a, b)$  is different from the pair  $(c, d)$ .

The Michigan Mathematics Prize Competition is an activity of the Michigan  
Section of the Mathematical Association of America.

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tion on the Approved List of Michigan Contests and Activities.