TWENTY-EIGHTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

PART I

October 10, 1984

INSTRUCTIONS

(to be read aloud to the students by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to insure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer. Usually a score of about 20 will allow you to become a finalist and write Part II of the competition.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "(E) none of these". If you believe none of the first four alternatives to be correct, mark E, in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request any one to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes, or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You may now open the test booklet and begin.

28TH ANNUAL MICHIGAN MATHEMATICS

PRIZE COMPETITION

#1 If x is a positive number such that $(x^2 - 1) \div (.04) = 8075$, then $x =$ _____.

a) 1.5 b) 323 c) 18 d) 449 e) none of these

#2 The smallest prime number greater than 100 is:

a) 101 b) 102 c) 103 d) 105 e) 107

#3 Which of the following lines has the largest positive slope?

a) $x - 10y = 3$ b) $x + 10y = 3$ c) $100x - y = 3$ d) $100x + y = 3$ e) $y = 400$

#4 Which of the following subsets of the real numbers is <u>not</u> the empty set?

a) $\{x | x^2 < 0\}$ b) $\{x | x = x + 1\}$ c) $\{x | x^2 + x + 1 = 0\}$ d) $\{x | \ |x| = \sqrt{-x}\}$

e) none of these

#5 If the whole number x has 3 zeros in its base two representation, then the number of zeros in the base two representation of $2x$ is:

a) 3 b) 5 c) 0 d) 4 e) none of these

#6 A <u>palindrome</u> is a number which reads the same forwards and backwards (for example, 767 and 111). How many palindromes fall between one thousand and ten thousand?

a) 100 b) 90 c) 81 d) 72 e) none of these

#7 If $x^2 = i$, where $i^2 = -1$, then

a) $\text{Re}\, x = -1$ b) x is purely imaginary c) $x = -1$ d) $|x| < 1$ e) none of these

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#8 Which one of the lines intersects each of the others in the x y plane?

a) $y = 3x - 7$ b) $y = 3x + 9$ c) $4x - y = 3$ d) $4x + y = 3$ e) $y = 4x - 12$

#9 The x-coordinate of the intersection of the graphs of $2x - y = -6$ and $x + y = -3$ is:

a) -9 b) -3 c) -2 d) 0 e) 3

#10 The length of a rectangle is 4 meters more than three times its width. If the perimeter of the rectangle is 56 meters, then its width is:

a) 6m b) 7m c) 8m d) 22m e) 28m

#11 If money in a bank quadruples every 10 years, then by what factor does it increase over a 30 year period?

a) 4 b) 8 c) 12 d) 16 e) 64

#12 The variable A is inversely proportional to the square of B . When A is ten, B is also ten. When B is one thousand, then A is:

a) 0.0001 b)0.001 c) 0.01 d) 100 e) 1000

#13 The sequence $N_0, N_1, N_2, N_3, \ldots$ is defined by $N_0 = 1$, $N_1 = -1$ and $N_i = N_{i-1} + 2\,N_{i-2}$ for $i \geq 2$. Then N_{100} is:

a) $2^{100} - 1$ b) 201 c) 101 d) 1 e) -1

#14 Twenty-seven white one-by-one-by-one cubes are stacked together to form a three-by-three-by-three cube (much like a Rubik^{TM} cube). All faces of the three-by-three-by-three cube are then painted red. How many faces of the original cubes remain white?

a) 243 b) 144 c) 122 d) 108 e) 99

#15 Which of the following is largest?

a) $3^{\log_3 9}$ b) $\sqrt{121}$ c) $\log_2 1024$ d) $\log_{\frac{1}{2}}(2^{-8})$ e) $\log_3 2187$

#16 The sum $1 + 2 + 4 + 8 + \ldots + 1024$ is:

a) 1041 b) 1040 c) 2049 d) 2048 e) 2047

#17 The infinite decimal $4.3\overline{26}$ (i.e., 4.3262626...) is equal to

a) $4 + \frac{326}{999}$ b) $4 + \frac{326}{1000}$ c) $4.3 + \frac{26}{99}$ d) $4.3 + \frac{26}{990}$ e) $\frac{4326}{999}$

#18 When $x^{26} - x^3 - 4$ is divided by $x + 1$, the remainder is:

a) -2 b) -1 c) 0 d) 1 e) 2

#19 The equation $a^{\log b} = b^{\log a}$, where the logarithm is to base 10, is:

a) True for all real numbers a and b

b) True only if $a = b$

c) Never true

d) True for all positive numbers a and b

e) Undefined unless both a and b are bigger than 10

#20 An expression equal to $\frac{3}{\sqrt{3} - \sqrt{6}}$ is:

a) $\frac{3}{\sqrt{3}} - \frac{3}{\sqrt{6}}$ b) $\frac{1}{\sqrt{2} - 1}$ c) $\frac{-3}{\sqrt{2}}$ d) $\sqrt{3} + \sqrt{6}$ e) $-(\sqrt{3} + \sqrt{6})$

#21 The hyperbolic tangent is defined by $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. The equation $\tanh(x) = 1$ has:

a) one real root b) infinitely many real roots c) one purely imaginary root

d) infinitely many complex roots e) no roots

#22 At a certain time between 4 o'clock and 5 o'clock, the minute hand lies on top of the hour hand. Sometime later, but before 5 o'clock, the minute hand is exactly opposite the hour hand. How much time elapses between the two events?

a) 30 min. b) 33 min. c) $\frac{240}{11}$ min. d) $\frac{600}{11}$ min. e) $\frac{360}{11}$ min.

#23 If $\log_3(x+1) = 2$, then $x =$

a) $\frac{2}{\log_{10}3} - 1$ b) 8 c) 7 d) 9 e) 10

#24 The graph of $y = -3x + 4$ is:

a) a vertical line b) a line rising to the right c) a horizontal line

d) a line falling to the right e) a line falling to the left

#25 If $\frac{(2x-1)(x-1)}{x-2} = 0$, then x is:

a) 1 or 2 b) $-\frac{1}{2}$ or -1 c) $\frac{1}{2}$ or 2 d) $\frac{1}{2}$ or 1 e) $-\frac{1}{2}$, -1 or -2

#26 The symbol "$\approx$" means "approximately equal to". Given that $a^6 \approx 500$, then $a^{12} \approx$

a) 1000 b) 250,000 c) 2,500,000 d) 25,000 e) $(500)^6$

#27 In the given figure, the distance between points A and C is:

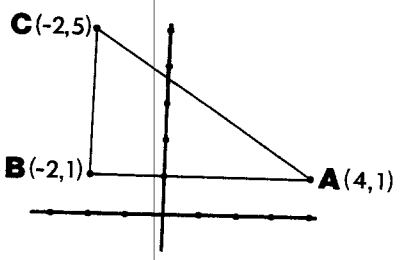
a) 7

b) 10

c) $2\sqrt{13}$

d) $2\sqrt{5}$

e) none of these



#28 $|x-2| \leq 1$ is equivalent to

a) $x \geq 3$ b) $x \leq 3$ c) $-3 \leq x \leq -1$ d) $1 \leq x \leq 3$ e) $-1 \leq x \leq 1$

#29 When simplified $(-\frac{1}{64})^{\frac{2}{3}}$ is

a) -16 b) $16\sqrt{-1}$ c) $\frac{1}{16}$ d) 16 e) $-\frac{1}{16}$

#30 The equation $\cos 2\theta = 2\cos\theta$

a) is an identity b) has no solutions c) is true only for $\theta = 0$

d) has a solution with $0 < \theta < \frac{\pi}{2}$, e) has a solution with $\frac{\pi}{2} < \theta < \pi$

#31 The highest power of 7 dividing 100! (= 100 · 99 · 98 · 97 ... 3 · 2 · 1) is:

a) 7^7 b) 7^{14} c) 7^{15} d) 7^{16} e) 7^{100}

#32 The graph of $x^2 = y^2 + 4y$ is:

a) a circle b) an ellipse but not a circle c) a parabola d) a hyperbola
e) none of these

#33 The sum of all the solutions to $4^{2x} + 2 = 3 \cdot 4^x$ is:

a) 1 b) 2 c) 3 d) 4 e) $\frac{1}{2}$

#34 Suppose A and B are sets of integers such that every element of set A is divisible by 2 and such that at least one element of set B is divisible by 3. Then

a) every element of $A \cap B$ is divisible by 3
b) some element of $A \cap B$ is divisible by 6
c) every element of $A \cup B$ is divisible by 3
d) some element of $A \cup B$ is divisible by 6
e) none of these

#35 The sum of five consecutive integers is 5. Then the smallest of the five integers is:

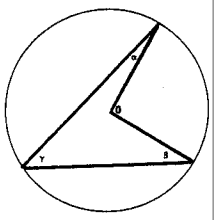
a) -3 b) -1 c) 0 d) 1 e) none of these

#36 If $27^n = 9^{2n+1}$, then n is:

a) -2 b) -1 c) 1 d) 3 e) none of these

#37 In the figure, O is the center of the circle, and $\gamma = 37°$. Then $\alpha + \beta$ is:

a) 37° b) 53° c) 74° d) 90° e) not enough information is given



#38 The number of solutions to the equation $\tan 4\theta = 1$ with $0 \leq \theta \leq 2\pi$ is:

a) none b) 1 c) 2 d) 4 e) 8

#39 The graph of $y = \frac{1}{1+x^2}$ has which of the following properties?

I. The graph lies entirely above the x-axis.
II. The graph is symmetric with respect to the y-axis.
III. The x-axis is an asymptote.

a) I only b) I and II only c) I and III only d) II and III only
e) I, II, and III

#40 What number is the output of the BASIC program listed below?

a) 0
b) 10
c) 110
d) 11
e) 55

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10  S = 0
20  FOR I = 1 TO 10
30  S = S + I
40  NEXT I
50  PRINT S
60  END
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The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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ACKNOWLEDGEMENTS

The following industries and professional organizations have provided generous financial support to this competition.

Burroughs Corporation
Michigan Bell Telephone
Michigan Council of Teachers of Mathematics
Kuhlman Corporation

The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.