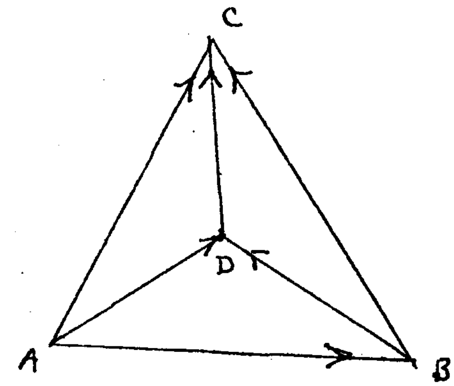
27 th, Part II

Problem 1. Find the largest integer which is a factor of all number of the form n(n + 1)(n +2) where n is an positive integer with unit digit 4. Prove your claims.

Solution: The smallest integer in the set under consideration is 4(5)(6) = 120, so the number we seek is 120 or less. The number must also have factors 5 (since n + 1 ends in 5), 3 (since one of 3 consecutive integers is divisible by 3) 8 (since n and n + 2 are consecutive even integers so one must contain a 4 factor). Thus the largest integer we seek has the form k (3) (5) (8) = 120 k $\leq$ 120 and it is thus 120.

Problem 2. Each pair of the towns A, B, C, D is joined by a single one way road. See example. Show that for any such arrangement a salesman can plan a route starting at an appropriate town that enables him to call on a customer in each of the towns.



Note that it is not required that he return to his starting point.

Solution 1. Considering for the moment only the 3 towns A, B, C it is clear that there is a route allowing a visit to each town. Without loss of generality this path can be taken to be $A \to B \to C$.

Now consider the roads in and out of D.

Case 1. $D \to A$.

Then $D \to A \to B \to C$ is the required path

Case 2. $D \leftarrow A$, $D \to B$

Then $A \to D \to B \to C$ is the required path

Case 3. $D \leftarrow A$, $D \leftarrow B$, $D \to C$

Then $A \to B \to D \to C$ is the required path

Case 4. $D \leftarrow A$, $D \leftarrow B$, $D \leftarrow C$

Then $A \to B \to C \to D$ is the required path

Solution 2. There can be at most one town with all roads leading in and at most one with all roads leading out. Hence there must be a town, say A, with two "out" roads and 1 "in" road or two "in" roads and 1 "out" road.

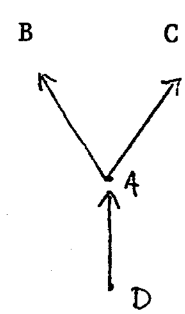
Case 1. $A \to B$, $A \to C$, $A \leftarrow D$

If $B \to C$, then the route is

$D \to A \to B \to C$.

If $B \leftarrow C$ then the route is

$D \to A \to C \to B$



Case 2. $A \to B$, $A \leftarrow C$, $A \leftarrow D$.

Argument is identical.

Problem 3. A and B are two points on a circular race track. One runner starts at A running counterclockwise and at the same time a second runner starts from B running clockwise.

They meet first 100 yds from A. measured along the track. They meet a second time at B and the third time at A. Assuming constant speeds how long is the track?

Solution. Denoting the respective rates by a and b the circumference of the track by c and the arc length from A counterclockwise to B as m we have :

$$\frac{a}{b} = \frac{100}{m-100} = \frac{m}{c} = \frac{c-m}{m}.$$

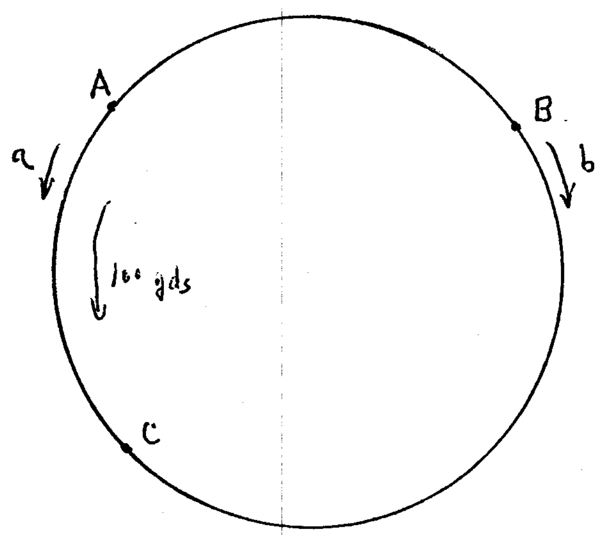
Hence $c = \dfrac{m(m-100)}{100}$ and $c = m + \dfrac{100m}{m-100}$

This leads to $m^2 - 300m + 100^2 = 0$ which has the roots

$$m = \frac{300 \pm 100\sqrt{5}}{2}$$ and since $m > 100$

$$m = 50(3 + \sqrt{5})$$

Thus $c = 100(2 + \sqrt{5}) = 423.6$ Approx.

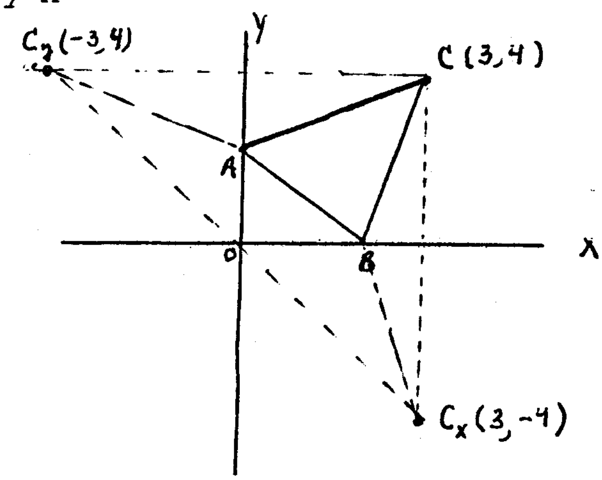


Problem 4 A and B are points on the positive x and positive y axes respectively and C is the point (3,4). Prove that the perimeter of Δ ABC is greater than 10.

Suggestion: Reflect ! !

Solution. Consider the points $C_x(3, -4)$ and $C_y(-3, 4)$ and observe that

$$\overline{CA} + \overline{AB} + \overline{BC} = \overline{C_yA} + \overline{AB} + \overline{BC_x} > \overline{C_yC_x} = \sqrt{36+64} = 10.$$



Problem 5. Let $A_1, A_2, \ldots A_8$ be a permutation of the integers $1, 2, \ldots, 8$ so chose that the eight sums $9 + A_1$, $10 + A_2$, $\ldots$, $16 + A_8$ and the eight differences $9 - A_1$, $10 - A_2$, $\ldots$, $16 - A_8$ together comprise 16 different numbers.

Show that the same property holds for the eight numbers in reverse order. That is, show that thé 16 numbers $9 + A_8$, $10 + A_7$, $\ldots$ $16 + A_1$ and $9 - A_8$, $10 - A_7$, $\ldots$ $16 - A_1$ are also pairwise different.

Solution. The given set of 16 numbers can be written:

$$\begin{matrix}(8 + i) + A_i \\ (8 + i) - A_i\end{matrix} \qquad i = 1, 2 \ldots, 8.$$

while the numbers in the second set are

$$\begin{matrix}(17 - i) + A_i \\ (17 - i) - A_i\end{matrix} \qquad i = 1, 2, \ldots, 8.$$

Now observe that

$$25 - ((17 - i) + A_i) = (8 + i) - A_i$$
$$25 - ((17 - i) - A_i) = (8 + i) + A_i$$

Thus any repetition in the second set would imply a repetition in the first set so the numbers in the second set must be distinct.