TWENTY-SIXTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

SOLUTIONS TO PART II

1. The key fact is that the distance in each direction for the pickup truck is the same. Let r be its rate going to get Sarah and let t be the time in hours that she spent walking. Then the truck spent $t - \frac{1}{4}$ hours going and $2 - t$ hours returning. Using "distance = rate times time" and equating for the truck's distance each way, we have

$$\begin{aligned} r(t - \tfrac{1}{4}) &= \tfrac{2}{3}r(2 - t) \\ t - \tfrac{1}{4} &= \tfrac{2}{3}(2 - t) \qquad \text{since clearly } r \neq 0 \\ \tfrac{5}{3}\cdot t &= \tfrac{19}{12} \\ t &= \tfrac{19}{20} \end{aligned}$$

Thus, Sarah walked for 19/20 hours or 57 minutes.

2. Solution 1 (Using vectors): Note that the vector $\overrightarrow{OB} = (-b,a)$ is perpendicular to $\overrightarrow{OA}$ and has the same length. Thus, clearly $\overrightarrow{MP} = \pm\frac{1}{2}\overrightarrow{OB} = \pm(-\frac{b}{2},\frac{a}{2})$. Then $(x,y) = \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = (\frac{a}{2},\frac{b}{2}) \pm (-\frac{b}{2},\frac{a}{2})$, so (x,y) is either $(\frac{a-b}{2},\frac{b+a}{2})$ or $(\frac{a+b}{2},\frac{b-a}{2})$.

Solution 2: Since MP is perpendicular to OA, the slope of MP is $\frac{-a}{b}$, so we have $(y-\frac{b}{2}) = -\frac{a}{b}(x-\frac{a}{2})$. Then

$$\begin{aligned} \overline{PM}^2 &= (x-\tfrac{a}{2})^2 + (y-\tfrac{b}{2})^2 = (x-\tfrac{a}{2})^2 + (\tfrac{a}{b})^2(x-\tfrac{a}{2})^2 = (x-\tfrac{a}{2})^2(1 + \tfrac{a^2}{b^2}) \\ &= (x-\tfrac{a}{2})^2(\tfrac{4}{b^2})(\tfrac{a^2+b^2}{4}) = \tfrac{4}{b^2}(x-\tfrac{a}{2})^2\overline{OM}^2 \end{aligned}$$

But since $\overline{PM} = \overline{OM}$, this yields $(4/b^2)(x-\frac{a}{2})^2 = 1$, from which we have $x = \frac{a}{2} \pm \frac{b}{2}$. Substituting this into the equation for MP yields $y = \mp\frac{a}{2} + \frac{b}{2}$. If $b = 0$, the above breaks down, but in this case, clearly $(x,y) = (\frac{a}{2},\pm\frac{a}{2})$, so the same formula holds.

3. Solution 1: Let S_n denote the sum of the first n terms. Begin by evaluation S_1, S_2, S_3, ... looking for a pattern:

$$S_1 = \frac{1}{1\cdot 2\cdot 3}$$

$$S_2 = S_1 + \frac{1}{2\cdot 3\cdot 4} = \frac{4+1}{2\cdot 3\cdot 4} = \frac{5}{2\cdot 3\cdot 4}$$

$$S_3 = S_2 + \frac{1}{3\cdot 4\cdot 5} = \frac{25+2}{2\cdot 3\cdot 4\cdot 5} = \frac{9}{2\cdot 4\cdot 5} = \frac{5+4}{2\cdot 4\cdot 5}$$

$$S_4 = S_3 + \frac{1}{4\cdot 5\cdot 6} = \frac{54+2}{2\cdot 4\cdot 5\cdot 6} = \frac{14}{2\cdot 5\cdot 6} = \frac{9+5}{2\cdot 5\cdot 6}$$

$$S_5 = S_4 + \frac{1}{5\cdot 6\cdot 7} = \frac{98+2}{2\cdot 5\cdot 6\cdot 7} = \frac{20}{2\cdot 6\cdot 7} = \frac{14+6}{2\cdot 6\cdot 7}$$

We note that it appears that

$$S_n = \frac{(1+2+\cdots+(n+1)) - 1}{2(n+1)(n+2)} = \frac{\frac{1}{2}(n+1)(n+2) - 1}{2(n+1)(n+2)} = \frac{(n+1)(n+2)-2}{4(n+1)(n+2)}$$

This formula can be verified by induction. Using it, we get

$$S_{98} = \frac{99\cdot 100 - 2}{4\cdot 99\cdot 100} = \frac{4949}{19800}$$

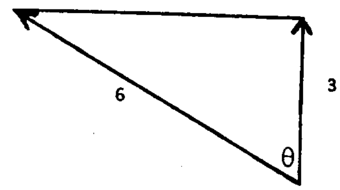
Solution 2: We use partial fractions methods to re-express $\frac{1}{n(n+1)(n+2)}$ in the form $\frac{\frac{1}{2}}{n} - \frac{1}{n+1} + \frac{\frac{1}{2}}{n+2}$. Then

$$S_{98} = \sum_{n=1}^{98}\left(\frac{\frac{1}{2}}{n} - \frac{1}{n+1} + \frac{\frac{1}{2}}{n+2}\right) = \sum_{n=1}^{98}\frac{\frac{1}{2}}{n} - \sum_{n=2}^{99}\frac{1}{n} + \sum_{n=3}^{100}\frac{\frac{1}{2}}{n}$$

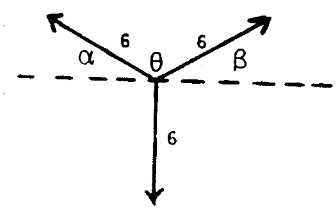
$$= \frac{\frac{1}{2}}{1} + \frac{\frac{1}{2}}{2} - \frac{1}{2} + \sum_{n=3}^{98}\left(\frac{\frac{1}{2}}{n} - \frac{1}{n} + \frac{\frac{1}{2}}{n}\right) + \left(-\frac{1}{99}\right) + \frac{\frac{1}{2}}{99} + \frac{\frac{1}{2}}{100}$$

$$= \frac{1}{4} + 0 - \frac{1}{99} + \frac{1}{2\cdot 99} + \frac{1}{2\cdot 100} = \frac{4949}{19800}$$

4. Solution 1 (Using physical intuition regarding symmetry): When the system reaches equilibrium, the tension in all parts of the cord will be six pounds. By symmetry, each of the segments AB and CB will be supporting three pounds in the vertical direction. Thus, we have the force vector diagram at the right for resolving the force in segment AB. By symmetry, θ is one half of angle ABC. Since $\theta = \arccos(3/6) = 60°$, angle ABC is 120°.



Solution 2 (With a more detailed analysis of force vectors): At equilibrium, the force vectors acting at B in the directions of the segments of the cord all have magnitude six and add up to zero. Thus, the sum of the horizontal and vertical components are zero. In the notation of the diagram, this yields the equations



(1) $6\cos\alpha - 6\cos\beta = 0$

(2) $6\sin\alpha + 6\sin\beta = 6$

Now (1) implies that $\cos\alpha = \cos\beta$, and since the cosine function is one-to-one for angles from 0° to 180°, this means that $\alpha = \beta$. Substituting this into (2), we get $\sin\alpha = \frac{1}{2}$, from which $\alpha = 30°$. Then angle ABC, which is θ, is 120°

Solution 3 (Outline): One can prove the same result based upon the principle from physics that at equilibrium, the system will have minimized its potential energy. That is, the weight will be as low as possible. One can express the vertical distance from the line AC to the weight in terms of several variables subject to a constraint and then use calculus methods to find the maximum for this distance. One finds that at the maximum, the angle ABC is 120°.

5. Part (a): Take four of the points p_i, p_j, p_k and p_m, and let Q denote the distance around the closed path from p_i to p_j to p_k to p_m and back to p_i. Let P be the common perimeter for all the triangles. Note that since $d_{ik} = d_{ki}$,

$$\begin{aligned} Q &= d_{ij} + d_{jk} + d_{km} + d_{mi} \\ &= (d_{ij} + d_{jk} + d_{ki}) + (d_{ik} + d_{km} + d_{mi}) - 2d_{ik} \\ &= 2P - 2d_{ik} , \end{aligned}$$

so $d_{ik} = P - \frac{1}{2}Q$. In exactly the same way, $d_{jm} = P - \frac{1}{2}Q$, so $d_{ik} = d_{jm}$.

We see that the distance between disjoint pairs of the points are equal. Similarly, distances between pairs with a point in common are also equal. Indeed, consider d_{ij} and d_{ik} with i, j and k all distinct. Let p_m and p_n be the points distinct from p_i, p_j and p_k. Then by the above result, d_{ij} and d_{ik} are both equal to d_{mn}, and hence are equal to each other. Thus, we have the distance between all pairs of points the same.

Part (b): No such set can exist in three dimensional space. Note that by the result in (a), if S were such a set, then p_1, p_2 and p_3 would form an equilateral triangle. The points p_4 and p_5, both being equidistant from these three, would each form with them a regular tetrahedron. There are two points in three dimensional space at which the fourth vertex of a regular tetrahedron with a given equilateral triangle as base can be located. Thus, p_4 and p_5 must be these points. But this makes $d_{45} = 2\sqrt{2/3}\cdot d_{12}$, which contradicts the result in (a).

It can be noted that such a set S can be found in a space of dimension four or higher.