TWENTY-SIXTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by
The Michigan Section of the Mathematical Association of America

PART 1

October 13, 1982

INSTRUCTIONS

(to be read aloud to the students by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to insure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in circles completely.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer. Usually a score of about 20 will allow you to become a finalist and write Part II of the competition.

5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "(e) none of these." If you believe none of the first four alternatives to be correct, mark E, in such cases.

6. No one is permitted to explain to you the meaning of any question. Do not request any one to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes, or any other aid is prohibited. If you have questions concerning the instructions, ask them now.

7. You may now open the test booklet and begin.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

26th ANNUAL MICHIGAN MATHEMATICS

PRIZE COMPETITION

1. Which of the following is true for all positive numbers A and B?

   (A) $\sqrt{A + B} = \sqrt{A} + \sqrt{B}$ (B) $\cos (A + B) = \cos A + \cos B$

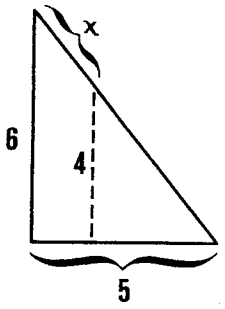
   (C) $\frac{1}{A} + \frac{1}{B} = \frac{1}{A + B}$ (D) $\log_{10}(A + B) = \log_{10} A + \log_{10} B$

   (E) none of these

2. $\left(\frac{1 + \sqrt{5}}{2}\right)^3 + \left(\frac{1 - \sqrt{5}}{2}\right)^3 =$

   (A) $2\sqrt{5}$ (B) 4 (C) $2(\sqrt{5})^3$ (D) $\frac{6\sqrt{5} + 2}{8}$ (E) none of these

3. Determine the length denoted x on the right triangle pictured.

   (A) $\frac{\sqrt{61}}{3}$ (B) $\sqrt{5}$ (C) $\frac{2\sqrt{61}}{3}$

   (D) $2\sqrt{2}$ (E) $\sqrt{61} - \sqrt{41}$

4. After 25 turns at bat, a baseball player had an average of .240. The next several games he improved and got 11 hits, and that raised his average to .340. How many times had he batted so far that whole season?

   (A) 25 (B) 1000 (C) 340 (D) 50 (E) 240

5. If $\log_e 2 = A$ and $\log_e 5 = B$, then $\log_e 1.6 =$

   (A) $4A$ (B) $\frac{3A}{B}$ (C) $3A - B$ (D) $A - B$ (E) $3A + 2B$

6. Give one pair (x,y) which is a solution of the equations $xy = -1$, $x + y = 1$:

(A) $x = -1,\ y = 1$ (B) $x = \frac{\sqrt{5} - 1}{2},\ y = \frac{\sqrt{5} + 1}{2}$

(C) $x = \frac{-1 + \sqrt{5}}{2},\ y = \frac{-1 - \sqrt{5}}{2}$ (D) $x = \frac{1 + \sqrt{5}}{2},\ y = \frac{1 - \sqrt{5}}{2}$

(E) no common solution

7. The line through (1,2) which is parallel to the line through the points (1,1) and (-2,5) has the equation

(A) $4x + 3y = 10$ (B) $4x + 3y = 11$ (C) $3y - 4x = 2$

(D) $3x + 4y = 11$ (E) none of these

8. An expression equal to $\frac{1}{\sqrt{3} + \sqrt{2}}$ is

(A) $\frac{1}{\sqrt{5}}$ (B) $\frac{\sqrt{3} - \sqrt{2}}{5}$ (C) $\frac{1}{\sqrt{6}}$ (D) $\sqrt{3} - \sqrt{2}$ (E) none of these

9. The function δ is defined on pairs of real numbers by the rule $\delta(x,y) = x^2 + xy$. What is $\delta(\delta(a,b), \delta(a, -b))$?

(A) $2a^4 + 2a^3b$ (B) $2a^4 + 2a^3b + 2a^2b^2$ (C) 0

(D) $a^4 - a^2b^2$ (E) none of these

10. When a bicycle with wheels one foot in radius rolls 100 feet, the number of times that the wheels turn, to the nearest whole number, is

(A) 8 (B) 16 (C) 32 (D) 100 (E) 200

11. If $\tan\theta = \frac{1}{2}$ and $\csc\theta < 0$, then $\cos\theta =$

(A) 2 (B) $\frac{2\sqrt{5}}{5}$ (C) $\frac{-2}{\sqrt{5}}$ (D) $\frac{-\sqrt{5}}{5}$ (E) $\frac{\sqrt{5}}{2}$

12. The set of all points P in a plane which are equidistant from a fixed point and a fixed line in that plane forms which of the following?

(A) circle (B) ellipse (C) prolapse (D) parabola (E) hyperbola

13. If I drive the first 3 miles at 45 mph, how fast must I drive the last mile in order to complete the entire trip in 5 minutes?

(A) 45 mph (B) $\frac{15}{4}$ mph (C) 1 mph (D) 60 mph (E) none of these

14. A factor of $x^8 + 64$ is

(A) $x + 2$ (B) $x^2 + 4$ (C) $x^4 - 4x^2 + 8$

(D) $x + 4$ (E) none of these

15. How many integers n are there with $1 \leq n \leq 4000$ which are multiples of 16 or 25 or both? (Regard an integer as a multiple of itself.)

(A) 160 (B) 250 (C) 400 (D) 410 (E) 420

16. How many quarts of a solution that is 80% water must be added to 10 qts. of a solution that is 40% water to produce a solution which is 64% water?

(A) 15 (B) 10 (C) 3 (D) 25 (E) can't be done

17. The smallest angle θ greater than 2520° which satisfies $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$ is

(A) 2550° (B) 2580° (C) 2640° (D) 2670° (E) none of these

18. In base 5 notation, the sum of 3042 and 1423 is

(A) 10020 (B) 4510 (C) 4505 (D) 4420 (E) none of these

19. If you put \$1,000 at the end of each year into a savings account earning interest at a rate r compounded annually, then after n years, your balance is given by the formula $\$1000\frac{(1+r)^n - 1}{r}$

The number of years it takes to accumulate \$50,000 is therefore given by the expression

(A) $\frac{50r+1}{r+1}$ (B) $\frac{\log(50r+1)}{\log(1+r)}$ (C) $\frac{-\log(50r)}{\log(1+r)}$

(D) 50 (E) none of these

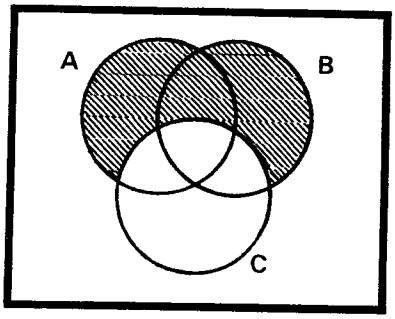
20. For the complex number $a + bi$ satisfying $(3+4i)(a+bi) = 3-4i$, the sum $a+b$ is

(A) $\frac{-31}{5}$ (B) -31 (C) $\frac{-31}{25}$ (D) 0 (E) 1

21. The infinite geometric series $5 + 5r + 5r^2 + 5r^3 + \ldots$ has a sum of 7. What is the common ratio r?

(A) $\frac{1}{2}$ (B) $\frac{2}{7}$ (C) $\frac{3}{7}$ (D) $\frac{5}{7}$ (E) none of these

22. Identify the set represented by the shaded region on the Venn diagram. (A bar over a set denotes its complement.)



(A) $((\overline{A} \cap B) \cup (A \cap \overline{B})) \cap \overline{C}$ (B) $(A \cup B) \cap \overline{C}$

(C) $(\overline{A} \cup \overline{B}) \cap \overline{C}$ (D) $(A \cup B) \cap C$

(E) $(A \cap B) \cup \overline{C}$

23. A cone has height h and base of radius r. At what distance from the apex must one cut the cone, parallel to the base, so that the pieces have equal volume?

(A) $\frac{h}{2}$ (B) $\frac{h}{\sqrt[3]{2}}$ (C) $\frac{r}{2}$ (D) $\frac{h}{(\sqrt[3]{2})^2}$ (E) none of these

24. Given that $\log_{10}3 = .4771$, determine how many digits there are in 3^{100}.

(A) 100 (B) 300 (C) 47 (D) 48 (E) none of these

25. Which of the following statements is true of the coefficient of x^{11} in the expansion of $(x - 2)^{16}$?

(A) It is positive.

(B) 2^5 is the highest power of 2 which divides it.

(C) 2^9 is the highest power of 2 which divides it.

(D) It is divisible by five different prime numbers.

(E) none of these.

26. The inequality $\frac{x + 1}{x^2 - 4} > 0$ is true for all real numbers x such that

(A) $x > 2$ or $x < -2$ (B) $x > 2$ or $-2 < x < -1$

(C) $-1 < x < 2$ (D) $x < -2$ or $x > -1$

(E) $x > 2$ or $x < -1$

27. For any number x, $\cos^4 x - \sin^4 x$ is equal to

(A) $\cos 2x$ (B) -1 (C) 1 (D) $\cot^4 x - \tan^4 x$ (E) $\cos 4x$

28. A certain marching band has a problem that when they march in rows of 4, there are 3 left over; when they march in rows of 5, there are 2 left over; and when they march in rows of 7, there are 4 left over. The smallest number of members that the band could have is

(A) between 30 and 39 (B) between 40 and 49 (C) between 50 and 59

(D) between 60 and 69 (E) between 70 and 79

29. The minimum of $f(x) = x + \frac{4}{x}$ for positive real numbers x is

(A) 5 (B) $\frac{13}{5}$ (C) $\frac{4}{3}$, but never reached

(D) $+\infty$ (E) none of these

30. Points A, B and C lie on a circle. The shorter arc between A and C contains one-third of the circumference and B is on the longer arc between them. What can be said about the angle ABC?

(A) It is a 60° angle. (B) It is a 120° angle.

(C) Its size depends on where B is, but it is between 90° and 180°.

(D) Its size depends on where B is, but it is between 0° and 90°.

(E) none of these

31. For the arithmetic series $3 + 8 + 13 + 18 + \ldots$, the sum of the first 200 terms is

(A) 2,150 (B) 20,318 (C) 40,725 (D) 80,510 (E) 100,100

32. How many sequences of four letters can be made using the 26 letters of the alphabet in such a way that no two adjacent letters are identical?

(A) 26^4 (B) $26^2 \cdot 25^2$ (C) $26 \cdot 25 \cdot 24 \cdot 23$ (D) $26 \cdot 25^3$ (E) none of these

33. Which combination of the following statements is true for the diagonals AC and BD of every rhombus $ABCD$?

(i) The diagonals bisect each other.

(ii) The diagonals intersect in a right angle.

(iii) The diagonals are equal.

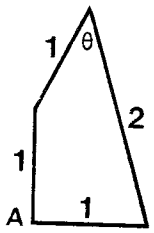
(A) all of them (B) only (i) (C) only (i) and (ii)

(D) only (i) and (iii) (E) none of these

34. The area of the largest triangle that can be put inside a semicircle of radius 5 with one side of the triangle on the diameter is

(A) 12.5 (B) 25 (C) 50 (D) 5π (E) none of these

35. What is the cosine of the angle θ in the quadrilateral pictured? The angle at A is a right angle.



(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$

(D) $\frac{\sqrt{3}}{4}$ (E) $\frac{2\sqrt{5}}{5}$

36. The circle which is the graph of the equation $x^2 - 2x + y^2 = 24$ has radius and center as follows:

(A) $2\sqrt{6}$ and (0,0) (B) $2\sqrt{6}$ and (-1,0) (C) 5 and (-1,0)

(D) 5 and (1,0) (E) 24 and (0,1)

37. If 6^{100} is divided by 7, the remainder is

(A) 6 (B) 0 (C) 1 (D) 2 (E) none of these

38. What is the highest power of 2 dividing $1^1 2^2 3^3 4^4 \ldots (15)^{15}$?

(A) 56 (B) 60 (C) 63 (D) 80 (E) none of these

39. If two roots of $ax^4 + bx + c = 0$ are -3 and 3, then b =

(A) 6 (B) -6 (C) $\frac{c}{2a}$ (D) 0 (E) none of these

40. If a 6, 7 or 8 shows up on a roll of a pair of dice, you win \$2; if anything else shows up, you lose \$1. If the dice are rolled 300 times, you expect to

(A) win \$600 (B) win \$300 (C) lose \$300

(D) lose \$150 (E) win \$100

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