

TWENTY-FOURTH ANNUAL  
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America

PART 1

October 15, 1980

INSTRUCTIONS

(to be read aloud to the students by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the answer sheet. Check to insure that your six-digit code number has been recorded correctly. Do not make calculations on the answer sheet. Fill in ovals completely.
2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheet.
3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.
4. You may be unfamiliar with some of the topics covered in this examination. You may skip over these and return to them later if you have time. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer. Usually a score of about 20 to 25 will allow you to become a finalist and write Part II of the competition.
5. In each of the questions, five different possible responses are provided. In some cases the fifth alternative is listed "(E) none of these." If you believe none of the first four alternatives to be correct, mark E, in such cases.
6. No one is permitted to explain to you the meaning of any question. Do not request any one to break the rules of the competition. The use of books, tables, slide rules, electronic calculators, notes, or any other aid is prohibited. If you have questions concerning the instructions, ask them now.
7. You may now open the test booklet and begin.

24th ANNUAL MICHIGAN MATHEMATICS

PRIZE COMPETITION

1. The number of real roots of the equation  $x^4 - 10x^2 - 1 = 0$  is

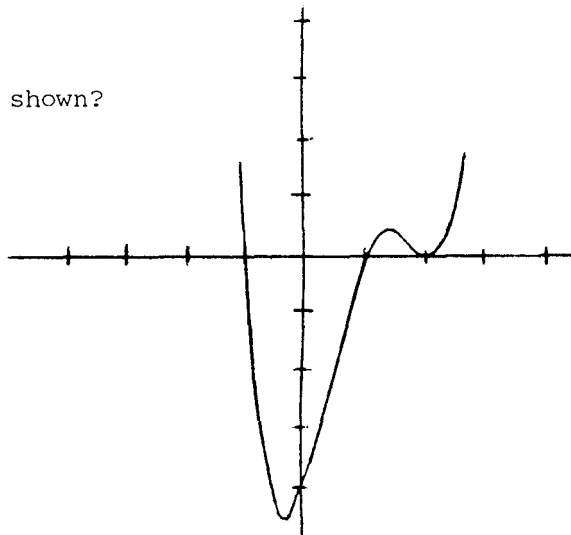
- A. 0      B. 1      C. 2      D. 3      E. 4

2. Another way to express  $(x-1)^{-2} - (x+1)^{-2}$  is

- A.  $4x(x^2-1)^{-2}$       B.  $2x(x^2-1)^{-2}$       C.  $-2$   
 D.  $2(x^2+1)(x^2-1)^{-2}$       E.  $-2^{-2}$

3. Which polynomial has its graph like the one shown?

- A.  $y = (x-1)(x+1)(x+2)$   
 B.  $y = (x^2-1)(x+2)^2$   
 C.  $y = (x^2-1)(x-2)^2$   
 D.  $y = (x+1)(x-1)(x-2)$   
 E. none of the above



4. A jewelry store marks some wrist watches at a certain price and then has a sale offering them at  $1/4$  off the marked price. The wholesale cost was only  $4/5$  of the final selling price. The ratio of the wholesale cost to the original marked price is

- A.  $5/6$       B.  $15/16$       C.  $1/5$       D.  $3/5$       E. 1

5. When  $x^{19} - 1$  is divided by  $x+1$  the remainder is

- A. 2      B. 0      C. 19      D. -2      E. -19

6. For all values of  $x$  for which the expression is defined,  $(\tan x + \cot x) \sin 2x$  equals

- A. 1      B. 2      C.  $2 \sec x$       D.  $2 \cos x + 2 \sin^2 x \sec x$   
 E. none of these

7. If a quadratic equation with real coefficients has no real roots, then the product of its roots is
- A. positive      B. negative      C. imaginary      D. any real number  $\neq 0$
- E. none of these
8. In rectangular coordinates, the equation  $y^2 - 16(x - 2)^2 = 0$  represents a
- A. parabola      B. hyperbola      C. point      D. circle      E. pair of straight lines
9. For what real numbers  $x$  is  $x^2 + \frac{1}{x} \geq 0$ ?
- A.  $x > 0$       B.  $x > 0$  and  $x \leq -1$       C.  $x \geq -1$
- D. all real  $x \neq 0$       E.  $x > 0$  or  $x \leq -1$
10. The number of water pipes with circular cross-section and having diameter 2 cm needed to carry the same volume of water as a single water pipe of diameter 12 cm is
- A. 216      B. 6      C. 12      D. 18      E. 36
11. If the diagonals of a quadrilateral  $Q$  bisect each other, then  $Q$  must be
- A. a square      B. a parallelogram      C. a rhombus      D. a rectangle
- E. none of these
12. An employee gets a pay raise of 50% and then in hard times has a pay cut of 50%. The net effect is
- A. a 10% raise      B. a 10% cut      C. a 25% raise      D. a 25% cut
- E. no change
13. A circle  $C$  of radius  $R - r$  lies entirely inside a circle  $C'$  of radius  $R + r$ , where  $0 < r < R$ . The area outside  $C$  and inside  $C'$  is
- A.  $\pi(R^2 - r^2)$       B.  $2\pi(R^2 + r^2)$       C.  $\pi(R + r)^2$       D.  $4\pi R r$
- E. cannot be determined
14. Taking logarithms to the base 2,  $\log 10 + \log 12 - \log 15$  equals
- A. 3      B. 8      C.  $\log 7$       D.  $2^7$       E. none of these

15.  $(\sin \frac{1}{x})^{-1}$  means the same as
- A.  $\csc x$       B.  $\sin x$       C.  $\arcsin x$       D.  $\sin^{-1}(\frac{1}{x})$       E. none of these
16. The number of positive integers less than one million which are neither perfect squares nor perfect cubes is
- A. 998,989      B. 998,901      C. 998,899      D. 998,910      E. none of these
17. At a party each person shakes hands exactly once with every other person. If the total number of handshakes was 45, how many were at the party? (When two people shake hands, it counts as one handshake).
- A. 9      B. 10      C. 15      D. 22      E. 90
18. An isosceles triangle has equal base angles of  $30^\circ$ . The vertex angle, in radian measure, is:
- A.  $\frac{3}{2}\pi$       B.  $\frac{3}{4}\pi$       C.  $\frac{5}{6}\pi$       D.  $\frac{120}{180}\pi$       E. none of these
19. Sally has an average grade of  $x$  on five French tests. She must take two more tests and desires her average on all seven tests to be  $y$  or better. Her last two tests must average at least
- A.  $7y - 5x$       B.  $\frac{1}{2}(6y - 5x)$       C.  $2y - x$       D.  $\frac{1}{2}(7y - 5x)$
- E. none of these
20. If  $(a + \frac{1}{a})^2 = 4$ , then  $|a^3 + \frac{1}{3}|$  equals
- A. 0      B. 2      C. 4      D.  $\sqrt{8}$       E. none of these
21. In base seven notation, a number which is 4 more than the base seven number 5064 is
- A. 5072      B. 5068      C. 20530      D. 5071      E. 5101
22. If  $\frac{1}{6}(6x^2 + 20x + 5m)$  is the square of a linear expression in  $x$ , then  $m$  lies between
- A. 3 and 4      B. 2 and 3      C. -1 and +1      D. -3 and -2      E. 1 and 2
23. A newly discovered large prime number is  $p = 2^{44497} - 1$ . The units digit of  $p$  is
- A. 1      B. 3      C. 5      D. 7      E. 9

24. Two vertical poles of heights 10 meters and 40 meters are 50 meters apart on level ground. A wire goes from the top of each pole to the foot of the other. The height in meters of the point of intersection of these wires is
- A. 7.2      B. 8      C. 16      D. 25      E. cannot be determined
25. The sum of the interior angles measured in radians of an  $n$ -sided regular polygon is
- A.  $\pi \cdot 2^{n-3}$       B.  $\frac{n(n-1)\pi}{6}$       D.  $(n-2)\pi$       D.  $(n+2)\pi$       E.  $2\pi/n$
26. Using three different weights, 1 lb., 2 lb., and 5 lb., how many objects of different weights can be weighed on a two pan balance scale?
- A. 13      B. 9      C. 8      D. 7      E. none of these
27. The total number of distinct positive integer divisors of  $p^2q^3$ , where  $p$  and  $q$  are distinct primes is
- A. 12      B. 10      C. 6      D. 5      E. none of these
28. Let  $S$  be the sum of odd integers with alternating signs going from 1 to 201:  
 $S = 1 - 3 + 5 - 7 + 9 - \dots + 201$ . Then  $S$  equals
- A. -99      B. -1      C. 1      D. 401      E. 101
29. A circle of radius 4 cm is inscribed in a triangle. The ratio of the perimeter of the triangle in cm to the area of the triangle in square cm is
- A.  $\frac{1}{2\sqrt{2}}$       B.  $\frac{\sqrt{2}}{2}$       C. 2      D.  $\frac{1}{2}$       E. none of these
30. There were  $M$  men and  $W$  women at a dance. If one of the men danced with 6 women, and a second man danced with 7 women, a third man with 8 women, etc., and the last man danced with all the women, then
- A.  $W = M + 6$       B.  $M = W - 5$       C.  $W = \frac{M(M+1)}{2}$
- D.  $W = M - 6$       E. none of these
31. If a bar denotes set complementation, then the complement of the set  $(A \cap \bar{B}) \cup (\bar{A} \cap B)$  is
- A.  $\bar{A} \cup B$       B.  $\bar{A} \cap \bar{B}$       C.  $\overline{(A \cup B)} \cup (A \cap B)$
- D.  $\overline{(A \cup B)} \cap \overline{(A \cap B)}$       E. none of these

32. If  $y = \frac{x^3}{(1+x)^n}$  and  $x > 0$ , then  $n$  equals

A.  $\frac{\frac{1}{3} \log y - \log x}{\log(1+x)}$

B.  $\frac{3 \log x - \log y}{\log(1+x)}$

C.  $\frac{\log y + \log(1+x)}{3 \log x}$

D.  $\frac{\log \left( \frac{y}{x^3} \right)}{\log(1+x)}$

E. none of these

33. The sum to infinity of the series

$$\frac{1}{5} + \frac{3}{5^2} + \frac{1}{5^3} + \frac{3}{5^4} + \frac{1}{5^5} + \frac{3}{5^6} + \dots \text{ is}$$

A.  $\frac{9}{25}$

B.  $\frac{1}{2}$

C.  $\frac{1}{3}$

D.  $\frac{2}{3}$

E. does not exist

34. A positive integer is called "lucky" if it is divisible by the sum of its digits, otherwise "unlucky". The smallest unlucky number divisible by 9 is

A. 99

B. 171

C. 189

D. 198

E. none of these

35. Team A won  $x$  games and lost  $y$ , where  $x$  and  $y$  are positive integers. Team B won one more game and lost one more game than team A. If the ratio of wins to losses is smaller for team A than for team B, then

A.  $x \geq y$

B.  $x = y$

C.  $x > y$

D.  $x < y$

E. none of these

36. A certain batch of fruit drink contains 5% fruit juice. A concentrate which is 80% fruit juice is added to make the mixture 10% fruit juice. What ratio of volume of concentrate to volume of the original batch is required?

A. 1 to 14

B. 2 to 5

C. 3 to 16

D. 3 to 14

E. none of these

37. Two evenly matched teams play a series of four games with each other. A game may end in a win, a loss, or a tie, and each outcome is equally likely. What is the probability that each team wins as many games as it loses (so series results in a tie)?

A.  $\frac{1}{2}$

B.  $\frac{2}{9}$

C.  $\frac{19}{81}$

D.  $\frac{13}{81}$

E.  $\frac{55}{81}$

38. If inflation causes prices to double every ten years, an article which now costs 1 dollar will cost  $2^{t/10}$  dollars in  $t$  years. The number of years required for prices to increase 100 fold is

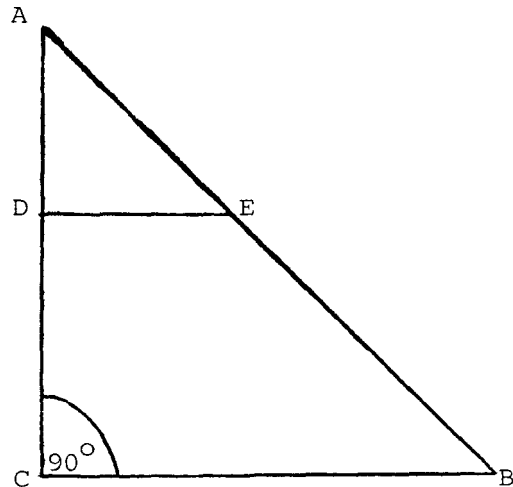
- A.  $\frac{20}{\log_{10} 2}$       B.  $10 \log_{10} 50$       C. 64      D. 50      E.  $\frac{100}{\log_{10} 20}$

39. For what real values of  $x$  does the equation  $y = \frac{\sqrt{x^2 + 2x}}{x + 1}$  yield real values for  $y$ ?

- A.  $0 \leq x \leq 1$       B.  $x \geq 2$  or  $x \leq 0$       C.  $x \leq -2$  or  $x \geq 0$   
D.  $x \neq -1$       E.  $-2 < x < 0$ , where  $x \neq -1$

40. In right triangle ACB, DE is parallel to CB. If  $AC = h$ ,  $CB = r$ ,  $DE = x$ ,  $DC = y$ , then

- A.  $y = \frac{h}{r}(r - x)$       B.  $y = \frac{h}{r} x$   
C.  $y = \frac{h}{r} \sqrt{r^2 - x^2}$       D.  $y = \frac{r}{h} x$   
E. none of these



The Michigan Mathematics Prize Competition is an activity of the Michigan  
Section of the Mathematical Association of America.

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tition on the Approved List of Michigan Contests and Activities.