PART II SOLUTIONS

1. Addition of the two equations gives \( \frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = \frac{5^2}{12^2} \).

Both sides are perfect squares and so \( \frac{1}{x} + \frac{1}{y} = \pm \frac{5}{12} \). Now the first equation is \( \frac{1}{x} \left( \frac{1}{x} + \frac{1}{y} \right) = \frac{1}{x} \left( \frac{\pm5}{12} \right) = \frac{1}{9} \) and so \( x = \pm\frac{15}{4} \).

Likewise the second equation is \( \frac{1}{y} \left( \frac{1}{x} + \frac{1}{y} \right) = \frac{1}{y} \left( \frac{\pm5}{12} \right) = \frac{1}{16} \) and so \( y = \pm\frac{20}{3} \). The corresponding solutions \((x,y)\) are \((\frac{15}{4}, \frac{20}{3})\) and \((-\frac{15}{4}, -\frac{20}{3})\).

Alternative: A somewhat less elegant solution follows by adding fractions to get \( \frac{x+y}{x^2y} = \frac{1}{9} \) and \( \frac{x+y}{xy^2} = \frac{1}{16} \).

Dividing the first of these by the second gives \( \frac{y}{x} = \frac{16}{9} \).

Multiplying the first equation by \( y^2 \) gives \( (\frac{y}{x})^2 + (\frac{y}{x}) = \frac{y^2}{9} \).

Finally, substituting \( \frac{y}{x} = \frac{16}{9} \) yields \( y^2 = 9 \left( \frac{256}{81} + \frac{16}{9} \right) = \frac{400}{9} \). Thus \( y = \pm\frac{20}{3} \).

Either substitution of this value or a similar approach of multiplying the second equation by \( x^2 \) will yield \( x = \pm\frac{15}{4} \).

2. Multiplying the given inequality by \( 30q \), gives the inequality \( 21q < 30p < 22q \). Direct computation of cases will show that \( q = 7 \) is the first value of \( q \) for which the interval \((21q, 22q)\) contains an integral multiple of 30. That is, \( 21 \cdot 7 = 147 < 30 \cdot 5 = 150 < 22 \cdot 7 = 154 \) and so \( p = 5 \) and \( q = 7 \) is the solution.

Alternative: From \( 21q < 30p < 22q \), we obtain \( 0 < 30p - 21q < q \).

The minimum positive linear combination is the gcd \((30, 21) = 3\).

By the Euclidean algorithm we obtain \( p = 5, q = 7 \). Thus \( \frac{5}{7} \) is the solution.
3. Since \( a_{n+1} = a_n^2 - a_n = a_n(a_n - 1) \), mathematical induction will show that \( a_{n+1} - 1 = a_na_{n-1} \cdots a_2a_1(a_1 - 1) \) for all \( n \geq 1 \). More informally,

\[
\begin{align*}
a_{n+1} - 1 &= a_n(a_n - 1) \\
&= a_n a_{n-1} (a_{n-1} - 1) \\
&\vdots \\
&\vdots \\
&= a_n a_{n-1} \cdots a_2a_1(a_1 - 1)
\end{align*}
\]

Further \( a_1 - 1 = 1 \) since \( a_1 = 2 \) is given. For any \( i > 1 \) we have \( a_i - 1 = a_i a_{i-1} \cdots a_2 a_1 - 1 \) and so \( a_i - 1 \) is divisible by every divisor of \( a_j \), \( j < i \). But this means that no divisor of \( a_j \) (other than \( \pm 1 \)) will divide \( a_i \) and so \( a_i \) and \( a_j \) are relatively prime.

Alternative: \( a_{i+1} - 1 = a_i^2 - a_i = a_i(a_i - 1) \). Thus \( a_i \) divides \( a_{i+1} - 1 \) and \( a_{i+1} - 1 \) divides \( a_i + 2 - 1 \). By induction and transitivity of divides, \( a_i \) divides \( a_j - 1 \) for \( j > i \).

So \( ka_i = a_j - 1 \) or \( a_j - ka_i = 1 \). Thus the gcd of \( a_i \) and \( a_j \) is 1.

4. Let \( n \) be the number of triangular regions and let \( e \) be the number of segments generated by the construction described. The \( n \) smaller triangles have \( 3n \) edges. Of these, 3 are the sides of the original triangle and the remaining \( 3n - 3 \) represents a double counting of the \( e \) segments, since each segment must be shared by two triangles. So \( 3n - 3 = 2e \) or \( e = 3(n - 1)/2 \). Since \( e \) is an integer, \( n - 1 \) must be even and hence \( n \) is odd.
5. Label the completed figure as shown at the right. Then by the Law Of Sines we have both
\[
\frac{x}{\sin 40^\circ} = \frac{x + y}{\sin 100^\circ} \quad \text{and} \quad \frac{x}{\sin (40^\circ - \theta)} = \frac{x + y}{\sin (40^\circ + \theta)}
\]
Thus \(\frac{x}{x + y} = \frac{\sin 40^\circ}{\sin 100^\circ}\) and \(\frac{x}{x + y} = \frac{\sin (40^\circ - \theta)}{\sin (40^\circ + \theta)}\)

So \(\frac{\sin 40^\circ}{\sin 100^\circ} = \frac{\sin (40^\circ - \theta)}{\sin (40^\circ + \theta)}\)

\(\sin (40^\circ) \sin (40^\circ + \theta) = \sin 100^\circ \sin (40^\circ - \theta)\)

\(\cos (50^\circ) \sin (40^\circ + \theta) = \sin 100^\circ \sin (40^\circ - \theta)\)

For \(\theta = 10^\circ\), this becomes \(\cos 50^\circ \sin 50^\circ = \sin 100^\circ \sin 30^\circ\)

and, since \(30^\circ = 1/2\), it reduces to \(2 \cos 50^\circ \sin 50^\circ = \sin 100^\circ\).

This is true from the double angle identity, hence \(\theta = 10^\circ\) satisfies the necessary law of sines. Since an acute angle solution must be unique, \(\theta = 10^\circ\) is the only solution.

Alternative: For a synthetic proof draw DE parallel to BC such that \(DE = BA\). Since \(DA = BC\) and \(\angle DEA = \angle ABC\) the triangles ABC and EDA are congruent. Therefore \(EA = AC\) and \(\angle DAE = 40^\circ\) and so \(\angle EAC = 60^\circ\), since \(\angle BAC = 100^\circ\). This implies that triangle ACE is equilateral and EC = ED and so triangle CED is isosceles. Thus \(\angle EDC = \angle ECD\). These base angles must sum to \(20^\circ\) since \(\angle CED = \angle AEC + \angle AED = 60^\circ + 100^\circ = 160^\circ\). Thus \(\angle EDC = 10^\circ = \angle BCD\), since they are alternate interior angles. (Other similar synthetic proofs exist.)