23rd

PART II SOLUTIONS

1. Addition of the two equations gives $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = \frac{5^2}{12^2}$.

Both sides are perfect squares and so $\frac{1}{x} + \frac{1}{y} = \pm\frac{5}{12}$. Now the first equation is $\frac{1}{x}(\frac{1}{x} + \frac{1}{y}) = \frac{1}{x}(\frac{\pm 5}{12}) = \frac{1}{9}$ and so $x = \frac{\pm 15}{4}$.

Likewise the second equation is $\frac{1}{y}(\frac{1}{x} + \frac{1}{y}) = \frac{1}{y}(\frac{\pm 5}{12}) = \frac{1}{16}$ and so $y = \pm\frac{20}{3}$. The corresponding solutions (x,y) are $(\frac{15}{4}, \frac{20}{3})$ and $(\frac{-15}{4}, \frac{-20}{3})$.

Alternative: A somewhat less elegant solution follows by adding fractions to get $\frac{x + y}{x^2 y} = \frac{1}{9}$ and $\frac{x + y}{xy^2} = \frac{1}{16}$.

Dividing the first of these by the second gives $\frac{y}{x} = \frac{16}{9}$.

Multiplying the first equation by y^2 gives $(\frac{y}{x})^2 + (\frac{y}{x}) = \frac{y^2}{9}$.

Finally, substituting $\frac{y}{x} = \frac{16}{9}$ yields

$y^2 = 9(\frac{256}{81} + \frac{16}{9}) = \frac{400}{9}$. Thus $y = \pm 20/3$.

Either substitution of this value or a similar approach of multiplying the second equation by x^2 will yield $x = \pm\frac{15}{4}$.

2. Multiplying the given inequality by 30 q, gives the inequality $21q < 30p < 22q$. Direct computation of cases will show that $q = 7$ is the first value of q for which the interval $(21q, 22q)$ contains an integral multiple of 30. That is, $21 \cdot 7 = 147 < 30 \cdot 5 = 150 < 22 \cdot 7 = 154$ and so $p = 5$ and $q = 7$ is the solution.

Alternative: From $21q < 30p < 22q$, we obtain $0 < 30p - 21q < q$. The minimum positive linear combination is the gcd $(30,21) = 3$. By the Euclidean algorithm we obtain $p = 5$, $q = 7$. Thus 5/7 is the solution.

3. Since $a_{n+1} = a_n^2 - a_n = a_n(a_n - 1)$, mathematical induction will show that $a_{n+1} - 1 = a_n a_{n-1} \cdots a_2 a_1(a_1 - 1)$ for all $n \geq 1$. More informally

$$
\begin{aligned}
a_{n+1} - 1 &= a_n(a_n - 1) \\
&= a_n\, a_{n-1}\, (a_{n-1} - 1) \\
&\;\vdots \\
&= a_n\, a_{n-1} \cdots a_2 a_1(a_1 - 1)
\end{aligned}
$$

Further $a_1 - 1 = 1$ since $a_1 = 2$ is given. For any $i > 1$ we have $a_i - 1 = a_1 a_2 \cdots a_{i-1}$ and so $a_i - 1$ is divisible by every divisor of a_j, $j < i$. But this means that no divisor of a_j (other than ± 1) will divide a_i and so a_i and a_j are relatively prime.

<u>Alternative</u>: $a_{i+1} - 1 = a_i^2 - a_i = a_i(a_i - 1)$. Thus a_i divides $a_{i+1} - 1$ and $a_{i+1} - 1$ divides $a_{i+2} - 1$. By induction and transitivity of divides, a_i divides $a_j - 1$ for $j > i$. So $ka_i = a_j - 1$ or $a_j - ka_i = 1$. Thus the gcd of a_i and a_j is 1.

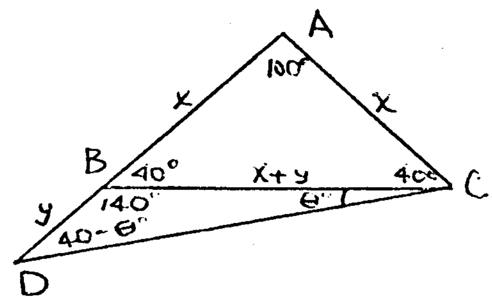
4. Let n be the number of triangular regions and let e be the number of segments generated by the construction described. The n smaller triangles have 3n edges.
Of these, 3 are the sides of the original triangle and the remaining $3n - 3$ represents a double counting of the e segments, since each segment must be shared by two triangles. So $3n - 3 = 2e$ or $e = 3(n - 1)/2$. Since e is an integer, $n - 1$ must be even and hence n is odd.

5. Label the completed figure as shown at the right. Then by the Law Of Sines we have both

$$\frac{x}{\sin 40^{\circ}} = \frac{x + y}{\sin 100^{\circ}} \quad \text{and} \quad \frac{x}{\sin (40^{\circ} - \theta)} = \frac{x + y}{\sin (40^{\circ} + \theta)}$$

Thus $\dfrac{x}{x + y} = \dfrac{\sin 40^{\circ}}{\sin 100^{\circ}}$ and $\dfrac{x}{x + y} = \dfrac{\sin (40^{\circ} - \theta)}{\sin (40^{\circ} + \theta)}$

So $\dfrac{\sin 40^{\circ}}{\sin 100^{\circ}} = \dfrac{\sin (40^{\circ} - \theta)}{\sin (40^{\circ} + \theta)}$



$$\sin (40^{\circ}) \sin (40^{\circ} + \theta) = \sin 100^{\circ} \sin (40^{\circ} - \theta)$$

$$\cos (50^{\circ}) \sin (40^{\circ} + \theta) = \sin 100^{\circ} \sin (40^{\circ} - \theta)$$

For $\theta = 10^{\circ}$, this becomes $\cos 50^{\circ} \sin 50^{\circ} = \sin 100^{\circ} \sin 30^{\circ}$ and, since $30^{\circ} = 1/2$, it reduces to $2 \cos 50^{\circ} \sin 50^{\circ} = \sin 100^{\circ}$. This is true from the double angle identity, hence $\theta = 10^{\circ}$ satisfies the necessary law of sines. Since an acute angle solution must be unique, $\theta = 10^{\circ}$ is the only solution.

Alternative: For a synthetic proof draw DE parallel to BC such that DE = BA. Since DA = BC and ∠EDA = ∠ABC the triangles ABC and EDA are congruent. Therefore EA = AC and ∠DAE = 40° and so ∠EAC = 60°, since ∠BAC = 100°. This implies that triangle ACE is equilateral and EC = ED and so triangle CED is isosceles. Thus ∠EDC = ∠ECD. These base angles must sum to 20° since ∠CED = ∠AEC + ∠AED = $60^{\circ} + 100^{\circ} = 160^{\circ}$. Thus ∠EDC = 10° = ∠BCD, since they are alternate interior angles. (Other similar synthetic proofs exist.)