TWENTY-THIRD ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America with the assistance of Michigan Colleges and Universities, Professional Organizations, and Industries.

PART 1

October 17, 1979

INSTRUCTIONS

(to be read aloud to class by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Check to insure that your six-digit student number has been recorded correctly. Do not make calculations on the answer sheet.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. The 40 problems of this examination are intended to sample many of the topics in the secondary mathematics curriculum. You may be unfamiliar with some of the topics covered. You may skip over these and return to them later if you have time. Usually a score of about 20 or more will allow you to become a finalist and write the second exam.

5. In each of the questions, five different possible responses are proposed. In some cases the fifth alternative is listed "(e) none of these." In such cases if you believe none of the first four alternatives to be correct, mark E.

6. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer or are able to eliminate some of the alternatives as impossible.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. The use of books, tables, slide rules, or electronic calculators is prohibited. If you have questions concerning the instructions, ask them now.

23rd ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

1. Given that $f(x) = 1 - \frac{1}{x}$ then $f\left(\frac{1}{x+1}\right)$ is

   (a) $1 - \frac{1}{x+1}$; (b) $\frac{x-1}{x(x+1)}$; (c) $\frac{x-1}{x+1}$; (d) $-x$; (e) none of these

2. The progression $a_1, a_2, \ldots, a_n, \ldots$ is geometric with positive terms and $a_5 = 4a_1$. Then the sum of the first five terms is

   (a) $5a_1$; (b) $10a_1$; (c) $(7 + 2\sqrt{3})a_1$; (d) $(3 + 7\sqrt{2})a_1$;

   (e) none of these

3. The equation $x^2 + bx + c$ has -2 for the sum of its roots and -24 for their product. The $b + c$ equals

   (a) -22 ; (b) -19 ; (c) 13 ; (d) 26 ; (e) none of these

4. Two spheres of unequal radii have a common center. If each radius is increased by 10% then the volume between the spheres is increased by

   (a) 10% ; (b) 20% ; (c) 33.1%; (d) 100% ; (e) none of these

5. If "p" people working "h" hours a day for "d" days produce abc articles, how many articles will "a" people working "b" hours a day produce in "c" days?

   (a) phd ; (b) $\frac{(phd)^2}{abc}$; (c) $\frac{(abc)^2}{phd}$; (d) $\frac{abc}{phd}$; (e) $\frac{phd}{abc}$

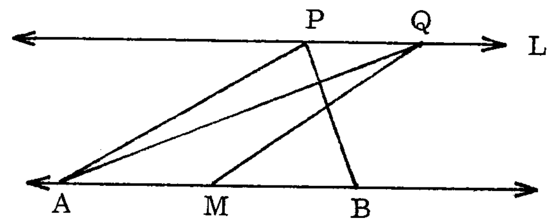
6. Let $\boxed{x} = x$ if x is even and $\boxed{x} = x + 1$ if x is odd. Given that $\boxed{x} = \boxed{y}$ then $x - y$ can only be

   (a) 0 ; (b) -1 or 0; (c) 0 or 1 ; (d) -1, 0 or 1; (e) -2, -1, 0, 1 or 2

7. Given the sequence $a_1 = 1$, $a_2 = 2$ and $a_n = a_{n-1} - a_{n-2}$ for all $n > 2$, then the sum of the first 100 terms is

(a) -1 ; (b) 4 ; (c) 2 ; (d) 1 ; (e) 3

8. In the diagram, M is the midpoint of AB and P and Q are distinct points on a line L parallel to the line containing AB. The ratio of the area of triangle APB to the area of triangle AQM is



(a) 2 ; (b) 4 ; (c) $\frac{1}{2}$; (d) insufficient information ; (e) none of these

9. The three sides of a triangle are 2, 3, and 4. The cosine of the greatest angle of the triangle is

(a) $\frac{3}{4}$; (b) $\frac{-\sqrt{3}}{2}$; (c) $\frac{4}{5}$; (d) $\frac{-1}{3}$; (e) $\frac{-1}{4}$

10. The trigonometric expression $2\cos^2\left(\frac{1+\theta}{2}\right)$ is identical to

(a) $1 + \cos(1+\theta)$; (b) $\cos(1+\theta) - 1$; (c) $1 - \cos(1+\theta)$

(d) $\cos^2(1+\theta)$; (e) none of these

11. Twenty distinct points are on a line. The number of distinct nonadjacent pairs of these points is

(a) 171 ; (b) 181 ; (c) 190 ; (d) 200 ; (e) none of these

12. The number of irrational roots for $x^4 - 6x^2 + 8$ is

(a) 0 ; (b) 1 ; (c) 2 ; (d) 3 ; (e) 4

13. I remarked, "Brothers and sisters I have none, but that man's father is my father's son." That man is my

(a) father ; (b) son ; (c) grandson ; (d) grandfather ; (e) cousin

14. One of the factors of $1 + x + x^2 + x^3 + x^4 + x^5 - x^6 - x^7 - x^8 - x^9$ is

(a) $1 + x + x^2$; (b) $1 + x + x^2 + x^3$; (c) $1 + x$; (d) $1 + x^2$; (e) none of these

15. A payroll is prepared by two computers in 6 hours. The faster computer can do the job by itself in 10 hours. In what time can the slower computer do the job?

(a) 12 hours; (b) 20 hours; (c) 15 hours; (d) 16 hours; (e) none of these

16. What values of x satisfy the inequality $\frac{|x - |x||}{x} > 0$?

(a) all $x \neq 0$; (b) all $x > 0$; (c) all $x < 0$; (d) no value of x; (e) none of these

17. If a, b, and c are nonzero real numbers then the range of

$f(a,b,c) = \frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{abc}{|abc|}$ is

(a) $\{-4, 0, 4\}$; (b) $\{4\}$; (c) all integers; (d) $\{0\}$; (e) none of these

18. F, G, and H are the three angles in a right triangle. Which of the following is always equal to $\sin^2 F + \sin^2 G + \sin^2 H$?

(a) $2 \sin^2 G$; (b) $2 \tan G \tan H$; (c) $2(\sin^2 F + \sin^2 H)$; (d) 2 (e) none of these

19. The graph of $y^2 = |1 - x^2|$

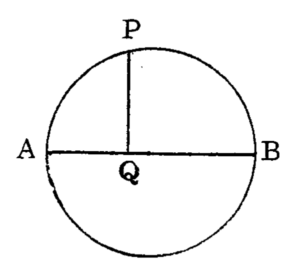
(a) is a hyperbola; (b) is a circle; (c) consists of a hyperbola and a circle;

(d) consists of a hyperbola and part of a circle; (e) none of these

20. If $\overline{A}$ represents the complement of set A then $\overline{(A \cup (\overline{B} \cap \overline{A}))}$ is the same as

(a) $\overline{A} \cup \overline{B}$; (b) $\overline{A} \cap \overline{B}$; (c) $A \cap \overline{B}$; (d) $\overline{A} \cap B$; (e) $\overline{A} \cup (B \cap A)$

21. AB is the diameter of a circle with radius 5 and PQ is perpendicular to AB at Q. If AQ = 2 then PB equals



(a) 8; (b) 9; (c) $3\sqrt{10}$ (d) $4\sqrt{5}$;

(e) none of these

22. The geometric mean of positive real numbers x and y is $G = \sqrt{xy}$ and their arithmetic mean is $A = (x + y)/2$. If $x \leq y$ and $A = 5$ and $G = 3$ then (x, y) is

(a) (5, 5) ; (b) $(\sqrt{15}, 4)$; (c) (9, 1) ; (d) (8, 2) ; (e) (1, 9)

23. The medians AD, BE and CF of an equilateral triangle ABC meet at point M. The midpoint of BD is X. If the area of triangle ABC is r times the area of triangle MBX, then r is

(a) 12 ; (b) 8 ; (c) 16 ; (d) 9 ; (e) 24

24. A square of maximum size is cut from a circular plate of radius R and then a circle of maximum size is cut from this square. The percentage of material wasted is

(a) $4\pi/25$; (b) $\pi R^2/200$; (c) 50; (d) $50/\sqrt{2}$; (e) none of these

25. If $\log_{16} A + \log_4 A = 4.5$, then A is

(a) 4 ; (b) 16 ; (c) 64 ; (d) 256; (e) none of these

26. The expression $\frac{x + 1}{x} - \frac{x}{x + 1}$ is the same as

(a) $\frac{1}{2x + 1}$; (b) $\frac{1}{x^2 + x}$ (c) $\frac{1}{x} - x$; (d) $\frac{x^2 + x + 1}{x^2 + x}$; (e) $\frac{1}{x} + \frac{1}{x + 1}$

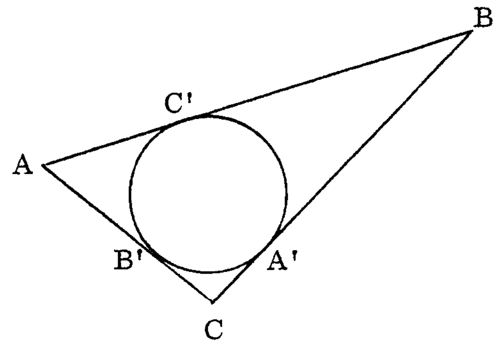
27. Let N(S) be the number of elements in set S. Let A - B be the set of elements in set A but not in set B. If $N(B) = 20$, $N(A \cap B) = 7$ and $N(A \cup B) = 55$ then $N(A - B)$ is

(a) 13 ; (b) 27 ; (c) 34 ; (d) 48 ; (e) none of these

28. If $a - b = 7$ and $ab = 60$ then $\sqrt{a^2 + b^2}$ is

(a) 13 ; (b) $\sqrt{71}$ (c) $\sqrt{74}$ (d) $\sqrt{129}$ (e) none of these

29. In triangle ABC the inscribed circle is tangent to the sides at A', B', and C'. Then in triangle A'B'C'

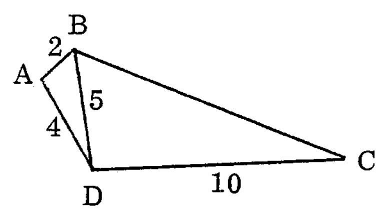


(a) exactly 2 angles are acute
(b) all angles are acute
(c) one angle is exactly 90^o.
(d) two angles are equal
(e) all three angles are equal

30. If $y = 1 + \frac{1}{3} \sqrt{36 - (x - 2)^2}$ then the maximum for y is

(a) 2 ; (b) 3 ; (c) 5; (d) -1 ; (e) 4

31. In the diagram, angles BAD and BDC are equal. The perimeter of ABCD is



(a) 89/4 ; (b) 57/2 ; (c) 30 ;

(d) 40 ; (e) 49/2

32. If $a^{\log_a 5} = a^4/125$ then a is

(a) 5 ; (b) 25 ; (c) 125 ; (d) 5 or 125 ; (e) 5 or 25

33. The sum $\frac{1}{2^2 - 1} + \frac{1}{3^2 - 1} + \frac{1}{4^2 - 1} + \frac{1}{5^2 - 1} + \ldots + \frac{1}{20^2 - 1}$ is

(a) $\frac{19}{399}$; (b) $\frac{38}{399}$; (c) $\frac{589}{840}$; (d) $\frac{589}{420}$; (e) $\frac{1}{2}$

34. A point P inside an equilateral triangle ABC, of side length 2, is at distances x, y, and z from sides BC, AC and AB respectively. The sum x + y + z is always

(a) between 0 and 1 ; (b) between 2 and 4 ; (c) $\sqrt{3}$

(d) rational ; (e) none of these

35. If $\log_{64} 289 = A$ and $\log_2 17 = B$ then B/A is

(a) 16 ; (b) 8 ; (c) 34 ; (d) 256 ; (e) 3

36. When 10^{21} is divided by 1002, the remainder is

(a) 8 ; (b) 42 ; (c) 874 ; (d) 998 ; (e) none of these

37. An isoceles trapezoid has bases of length a and b with $a < b$. The line joining the midpoints of the two equal legs divides the interior of the trapezoid into areas A_1 and A_2 with $A_1 < A_2$. Then the ratio A_1/A_2 is

(a) $\frac{2a}{a+b}$; (b) $\frac{3a+b}{3b+a}$; (c) $\frac{2b}{a+b}$; (d) $\frac{b+2a}{a+2b}$; (e) none of these

38. If $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$ and $x > 0$, then x must be

(a) 2^2 ; (b) $\sqrt{2}^{\sqrt{2}}$; (c) $\log_2 e$; (d) $\log_e 2$; (e) $\sqrt{2}$

39. If p and q are integers and $.818181\ldots = p/q$, then pq could equal

(a) 99 ; (b) 91 ; (c) 297 ; (d) 8100 ; (e) none of these

40. The number $\frac{\sqrt{\sqrt{5}+2} + \sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}$ is equal to

(a) $\sqrt{2}+1$; (b) $\sqrt{2}$; (c) $\sqrt{5}$; (d) $\sqrt{5}+1$; (e) $\sqrt{2}+\sqrt{5}$

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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J. K. Bidwell
Central Michigan University

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