SOLUTIONS

1. Let the fold line be F_1F_2 and let CO be $\perp$ to F_1F_2.

   After folding, AO is superimposed on CO, and hence $AO \perp F_1F_2$ also.

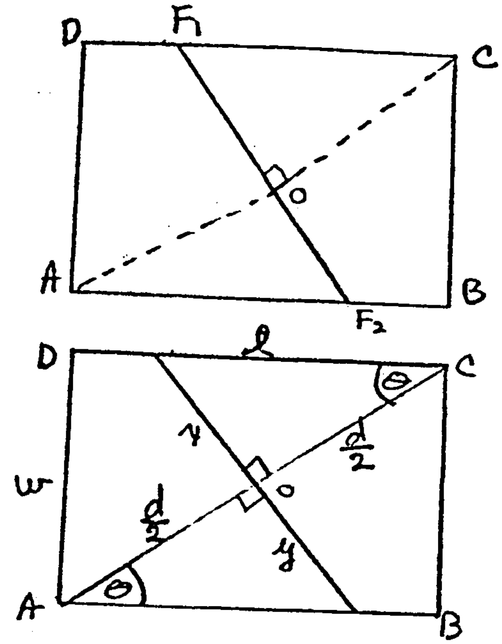
   Thus, AOC is a straight line, and it is bisected by F_1F_2. Let d be its length.

   Using ΔADC yields

   $$d^2 = \ell^2 + w^2 .$$

   Since ΔCOF_1 and ΔAOF_2 are similar to ΔADC,

   $$\frac{x}{d/2} = \frac{w}{\ell} = \frac{y}{d/2} .$$



   Thus, length of fold $= x + y = \frac{d}{2}\frac{w}{\ell} + \frac{d}{2}\frac{w}{\ell} = \frac{w}{\ell} d = \frac{w}{\ell}\sqrt{w^2 + \ell^2}$

2. Clearly, 1, 2, 4, and 7 are not possible. 3, 6 = 3 + 3, 8 = 3 + 5, and 9 = 3 + 3 + 3 are possible. The proof that all larger amounts are possible proceeds by induction (starting at 9).

   Assume $n \geq 9$ is possible. Then $n = 3k + 5j$, where k and j are non-negative integers.

   Case I $\quad j \neq 0$. Then

   $$\begin{aligned} n + 1 &= (3k + 5j) + 1 \\ &= 3k + 5(j - 1) + 6 \\ &= 3(k + 2) + 5(j - 1) . \end{aligned}$$

   Case II $\quad j = 0$. Then $k \geq 3$ since $n \geq 9$.

   $$\begin{aligned} n + 1 &= (3k + 0) + 1 \\ &= 3(k - 3) + 9 + 1 \\ &= 3(k - 3) + 5 \cdot 2 . \end{aligned}$$

   Thus, if n is possible, so is $n + 1$. Hence all $n \geq 9$ are possible.

3. Relabel the objects, if necessary, so that

$$0 \le w_1 \le w_2 \le \dots \le w_{21} .$$

Then,

$$W = w_1 + \dots + w_{10} \le w_{11} + \dots + w_{20} = \overline{W} .$$

Equality is possible iff $w_1 = w_2 = \dots = w_{20}$.

Any subset of 10 or 11 of the last 11 objects has weight greater than or equal to $\overline{W}$. Therefore, to match the total weight W of the first 10 objects, one must have

$$\overline{W} = W \quad \text{and} \quad w_1 = w_2 = \dots = w_{20} = x .$$

Consider $w_{11} + w_{12} + \dots + w_{20} + w_{21} = 9x + w_{21}$. This total weight must be matched by taking either 10 or 11 of the first 11 objects. Hence, either

$$10x = 9x + w_{21} \qquad \text{or} \qquad 11x = 9x + w_{21} .$$

In the first case, $w_{21} = x$, and in the second case, $w_{21} = 2x$.

Therefore, either

i) all 21 objects have the same weight, or

ii) 20 have the same weight w and one object is twice as heavy as the others.

4. Let $x^3 + x^2 - 1 = P(x) = (x - r)(x - s)(x - t)$.

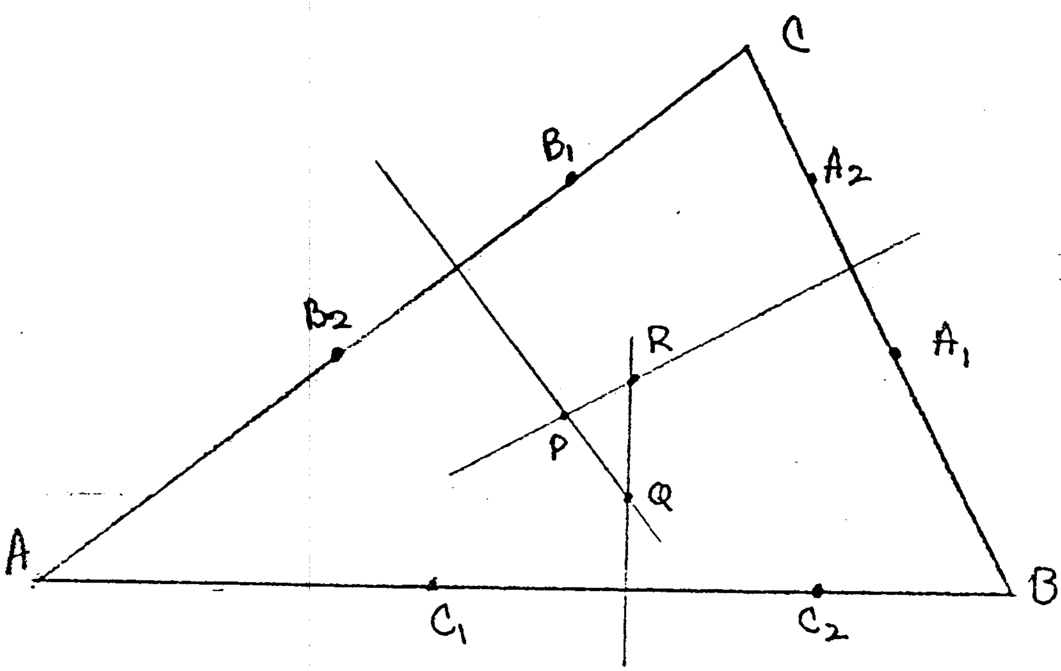
$P(t) = 0$ and $1 = rst$. Hence, $t \ne 0$ and $rs = 1/t$.

$$Q(rs) = Q\left(\frac{1}{t}\right) = \left(\frac{1}{t}\right)^3 - \left(\frac{1}{t}\right) - 1$$

$$= \frac{1 - t^2 - t^3}{t^3} = \frac{P(t)}{t^3} = \frac{-0}{t^3} = 0 .$$

5. Let the perpendicular bisectors of A_1A_2, B_1B_2, and C_1C_2 be, respectively, L_A, L_B, L_C. L_A and L_B meet at P, the center of the circle passing through A_1, A_2, B_1, and B_2; L_B and L_C meet at Q, the center of the circle passing through B_1, B_2, C_1, and C_2; L_C and L_A meet at R, the center of the circle passing through C_1, C_2, A_1, and A_2.

<u>Case I</u> If $P = Q = R$, then this is the center of a circle passing through all six points.



<u>Case II</u> The lines L_A, L_B, and L_C form a triangle, PQR inside the triangle ABC. Note: In the figure, the vertices P, Q, and R could occur in clockwise rather than counter-clockwise order, but the argument is similar in that case.

Let $x = PA_1 = PA_2 = PB_1 = PB_2$,
$y = QB_1 = QB_2 = QC_1 = QC_2$, and
$z = RC_1 = RC_2 = RA_1 = RA_2$.

$x = PB_2 < QB_2 = y$
$y = QC_2 < RC_2 = z$
$z = RA_2 < PA_2 = x$

Hence $x < y < z < x$, a contradiction. Thus, Case I is the only possible case.