21st ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Sample Solutions for Part II

1. A teenager coming home after midnight heard the hall clock striking the hour. At some moment between 15 and 20 minutes later, the minute hand hid the hour hand. To the nearest second, what time was it then?

Solution: t is the time in minutes between the time that the clock struck the hour and the moment between fifteen and twenty minutes later when the minute hand covered the hour hand. The minute hand was between the numerals 3 and 4 since $15 \leq t \leq 20$, $3 \leq t/5 \leq 4$. The hour struck must be 3. Also, the arc of the hour hand measured from 12 will be $1/4 + t/60 \cdot 1/12$. Hence, $t/60 = 1/4 + t/60 \cdot 1/12$, $12t = 180 + t$, and $t = 16\ 4/11$ min = 16 min 22 sec to the nearest second. The time was 3:16:22.

Alternate Solution: Let t be the number of minute spaces the hour hand moved from struck hour to coincidence with minute hands. Since the struck hour must be 3:00 a.m., $15 + t = 12t$ (since the minute hand goes 12 times as fast as the hour hand). Thus $t = 1\ 4/11$ min = 1 min 22 sec approximately, so that the time in question is 3 hours, 16 minutes and 22 seconds.

2. The ratio of two positive integers a and b is 2/7, and their sum is a four digit number which is a perfect cube. Find all such integer pairs.

Solution: Let $a = 2x$, $b = 7x$; so $a + b = 9x = c^3$, and $x = 3D^3$ where $1000 \leq 27D^3 \leq 9999$ and $37 < D^3 < 370$. Thus $D = 4, 5, 6$ or 7, and $D^3 = 64, 125, 216, 343$, and $x = 3D^3 = 192, 375, 648, 1029$. Thus $a = 2x$, $b = 7x$ yeilds:

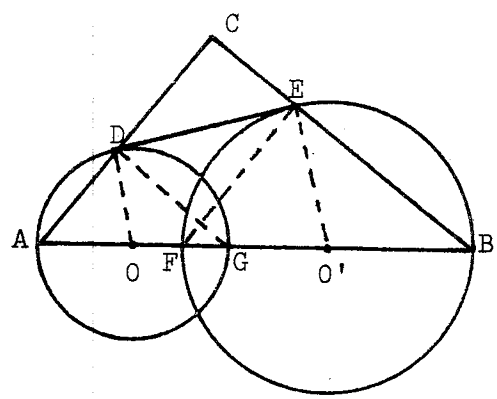
384	750	1296	2058
1344	2625	4536	7203
1728	3375	5832	9261
"	"	"	"
12^3	15^3	18^3	21^3

3. Given the integers $1, 2, \ldots, n$, how many distinct numbers are of the form $\sum_{k=1}^{n} (\pm k)$, where the sign $(\pm)$ may be chosen as desired? Express answer as a function of n. For example, if $n = 5$, then we may form numbers:

$$1 + 2 + 3 - 4 + 5 = 7$$
$$-1 + 2 - 3 - 4 + 5 = -1$$
$$1 + 2 + 3 + 4 + 5 = 15 \text{ , etc.}$$

Solution: The largest and smallest numbers obtainable are $1 + \ldots + n = \frac{n(n+1)}{2}$, $-1 -2 \ldots -n = -\frac{n(n+1)}{2}$. Thus $n(n+1) + 1$ is the most distinct numbers possible. We show that only every other number can be attained, of which there are $\frac{n(n+1)+1+1}{2} = \frac{n(n+1)}{2} + 1$. First, all numbers that can be attained have the same parity since $\sum_1^n(\pm)k - \sum_1^n(\pm)^* k = \sum_1^n 2(\frac{+}{0})k$ where $(\pm)$ indicates a sign choice, $(\pm)^*$ a different sign choice and $(\frac{+}{0})$ indicates a choice of +1, −1, 0. The final step is by induction. Let $\sum(\pm)k$ be any number other than $\frac{(n)(n+1)}{2}$. If the representation has a term with a plus sign followed by a term with a minus sign, say j, $-(j+1)$, reversal of just these signs increases the number by 2. If there is no plus sign followed by a minus sign, then the first term must be -1 and changing this to $+1$ increases the sum by 2. To summarize, if $N < \frac{n(n+1)}{2}$ is attained, so is $N + 2$, and since $-(\frac{n(n+1)}{2})$ is attained, every other number in the range $[-(\frac{n(n+1)}{2}), \frac{n(n+1)}{2}]$ is attained and only these. Hence $\frac{n(n+1)}{2} + 1$ is the answer.

4. $\overline{DE}$ is a common external tangent to two intersecting circles with centers at 0 and 0'. Prove that the lines $\overline{AD}$ and $\overline{BE}$ are perpendicular.



Solution: Draw $\overline{DG}$, $\overline{FE}$. Use facts (1) angle subtending semi circle is 90°; (2) angle between tangent and chord measured by half intercepted arc; (3) inscribed angle measured by half of arc. Add angles in quadrilateral ADEB $2\alpha + 90° + 2\beta + 90° = 360°$; $\alpha + \beta = 90°$. Hence, in ΔABC, $\angle C$ is 90°.

Alternate Solution: Draw $\overline{DO}$ and $\overline{EO'}$. Use facts (1) base angles of isosceles triangle are equal; (2) radius drawn to point of contact to tangent line is perpendicular to tangent line; (3) sum of interior angles of quadrilateral is 360°.

$\angle OAD + \angle ADO + \angle ODE + \angle DEO' + \angle O'EB + \angle EBO' = \alpha + \alpha + 90° + 90° + \beta + \beta = 360°$

Hence in ΔABC $\alpha + \beta = 90°$ and $ACB = 90°$.

5. Find all polynomials $f(x)$ such that $(x-2)\, f(x+1) - (x+1)\, f(x) = 0$ for all x.

Solution: Given $(x-2)\, f(x+1) - (x+1)\, f(x) \equiv 0$. Note that $f(2) = 0$, $f(0) = 0$, $f(1) = 0$, so that $f(x) = (x-2)(x)(x-1)\, g(x)$. We now have:

$(x-2)[(x-1)(x+1)(x)\, g(x+1)] - (x+1)[(x-2)(x)(x-1)\, g(x)] \equiv 0$, or

$x(x+1)(x-1)(x-2)[g(x+1) - g(x)] \equiv 0$, so that

$g(x+1) = g(x)$ for all x, hence $g(x)$ is a constant and $f(x) = k\, x(x-1)(x-2)$.