TWENTY-FIRST ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America with the assistance of Michigan Colleges and Universities, Professional Organizations, and Industries.

PART 1

October 19, 1977

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Check to insure that your six-digit student number has been recorded correctly. Do not make calculations on the answer sheet.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease to work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgements. For each problem you should work out ideas on scratch paper before selecting the answer.

4. The first 20 problems of this examination are intended to sample many of the topics in the secondary mathematics curriculum. You may be unfamiliar with some of these topics and quite possibly will find a number of problems which are easier for you distributed throughout the last twenty items. Usually a score of about 20 or more will allow you to become a finalist and write the second exam.

5. In each of the questions five different possible responses are proposed. In some cases the fifth alternative is listed "E, none of these". In such cases if you believe none of the first four alternatives to be correct, mark E.

6. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer or are able to eliminate some of the alternatives as impossible.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. The use of books, tables, slide rules or electronic calculators is prohibited. If you have questions concerning the instructions ask them now.

# 21st ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

1. A, B, and C are sets, $A \cup B = \{1,2,3,4\}$ , $A \cup C = \{1,2,3,4,5,6\}$ , $A \cap B = \{2,4\}$ , $A \cap C = \phi$ , and $B \cap C = \{1,3\}$ . Find $B$ .

   A. $B = \{1,2,3,4\}$ B. $B = \{1,3,5,6\}$ C. $B = \{2,4,5,6\}$

   D. $B = \{2,4\}$ E. $B = \{5,6\}$

2. How many roots of $x^4 - 9x^2 + 20 = 0$ are rational?

   A. 0 B. 1 C. 3 D. 2 E. 4

3. One leg of a right triangle is increased by 10% and the other leg is decreased by 10% . The area of the new right triangle is

   A. 1% smaller B. 1% larger C. 5% larger

   D. 10% larger E. unaltered

4. If each hen can lay 2 eggs in 3 days, how many days are needed for 4 hens together to lay 2 dozen eggs?

   A. 12 B. 6 C. 5 D. 8 E. 9

5. A ball is dropped from a height of 18 feet onto a cement sidewalk. Each time it bounces up it goes exactly one third as high as the time before. The total distance the ball travels in feet is

   A. 36 B. 27 C. 45 D. 54 E. infinite

6. In a basketball freethrow contest, a man gets a dime every time he makes a basket, and pays a quarter for each missed shot. If he takes 100 shots and wins \$5.10, how many times did he miss?

A. 14 B. 34 C. 68 D. 50 E. 86

7. The curves $x = y^2$ and $x^2 + (y-1)^2 = 1$ intersect at how many points?

A. 0 B. 1 C. 3 D. 2 E. 4

8. Find all real numbers x which satisfy the inequality $3 \leq |x-1| \leq 7$.

A. $4 \leq x \leq 8$ or $-8 \leq x \leq -4$ B. $x \leq -2$ or $x \geq 4$

C. $-2 \leq x \leq 4$ D. $-6 \leq x \leq -2$ or $4 \leq x \leq 8$

E. $-6 \leq x \leq 8$

9. The number 597 is to be written in a base 5 numeration system. That is, $597 = a_1 + a_2(5) + a_3(5^2) + \ldots a_n(5^n)$, where the $a_i$'s are non-negative integers $0 \leq a_i \leq 4$. Find $a_2$.

A. 0 B. 1 C. 2 D. 3 E. 4

10. If $\log_{10}2 = .3010$ and $\log_{10}3 = .4771$, then $\log_{10}5 = ?$

A. .7781 B. .6990 C. .5229 D. .5168

E. Impossible to determine with given information

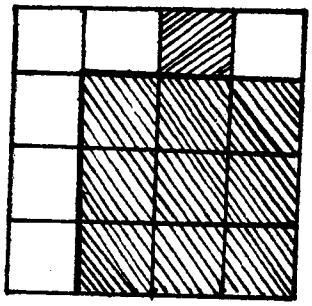
11. Let $f(x) = x^2$ and $g(x) = 4x^2 + a$. For what values of a will the graphs of f and g intersect?

A. all a B. $a \geq 0$ C. $a \leq 0$ D. $a \geq 4$ E. None of these

12. Suppose each year we use up half of what remains of some resource. To the nearest year, how many years will it take to use up 99.9% of what we now have of this resource?

A. 2 B. 10 C. 49 D. 110 E. 128

13. What is the total number of squares of various sizes that may be formed in the accompanying figure? (Two squares are shown.)



A. 16 B. 30 C. 36 D. 64 E. 100

14. A cube has volume 1. What is its surface area?

A. 1 B. 3 C. 2 D. $\sqrt{3}$ E. 6

15. Which of the following should be deleted so that the remaining four have equal value?

A. $16/27$ B. $4^2/3^3$ C. $3^2/2^4$ D. $2(2/3)^3$ E. $144/3^5$

16. Which of the following is the largest? (Logs are to base 10).

A. 10 B. $\log 10^{11}$ C. $1000^{1/3}$ D. 3 log 100 E. $(\frac{1}{3})^{-1/3}$

17. You are presented with a system of three linear equations in three unknowns. How many distinct solutions will this system of equations have?

A. 0 B. 1 C. 3 D. An infinite number
E. Not enough information

18. If $f(x) = \sqrt{x+1} + 9x^{-2} + 1$, then $f(3)$ equals

A. 0 B. 3 C. 4 D. 6 E. 12

19. The product of the roots of the equation $3x^2 + 9x = 2$ is equal to

A. 6 B. -3 C. -2 D. -2/3 E. None of these

20. The negation of the statement "Some students do not like geography" is the statement:

A. All students do not like geography.

B. Geography likes all students.

C. All students like geography.

D. No student likes geography.

E. None of the previous.

21. A small circle of radius $r$ lies inside a larger circle of radius $R$ and is tangent to the larger circle and to two perpendicular diameters of the larger circle. If $R = \sqrt{2}$, then $r$ equals

A. 1 B. $\frac{\sqrt{2}}{2}$ C. $2 - \sqrt{2}$ D. $\sqrt{2} - 1$ E. $1/\sqrt{2}$

22. Let $N(A)$ denote the number of elements in a set $A$. Twenty horses are stabled in a barn containing 30 stalls. If $H$ is the set of horses, and $S$ the set of stalls, then

A. $H \subset S$ B. $S \subset H$ C. $H \cup S = 50$

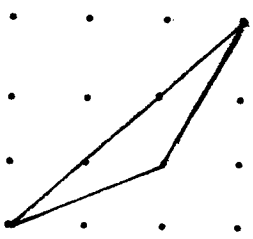
D. $N(H \cap S) = 20$ E. None of these

23. Find all values of $x$ for which $\log_a(x^2-5x+7) \geq 0$, where $a > 1$.

A. $x \geq 3$ or $x \leq 2$ B. $2 \leq x \leq 3$ C. $x \leq -1$

D. $\frac{5-\sqrt{3}}{2} \leq x \leq \frac{5+\sqrt{3}}{2}$ E. The statement is true for all values of $x$.

24. Find the area of the triangle. Horizontal or vertical distance between consecutive dots is one unit. Answers are in square units.

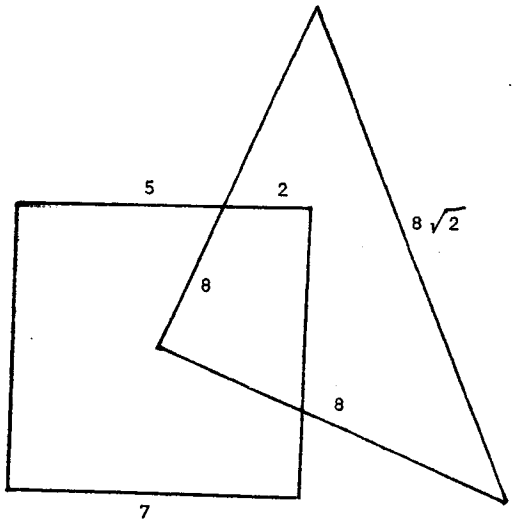


A. $\frac{\sqrt{2}}{2}$ B. $\frac{\sqrt{3}}{2}$ C. $\frac{3}{2}$ D. $\frac{5}{2}$ E. 4

25. The value of $\sum_{1}^{51} (-1)^n (2n-1) = -1 + 3 - 5 + 7 \ldots + 99 - 101$ is

A. 2 B. -2 C. 0 D. -76 E. -51

26. An isosceles right triangle whose equal sides have length 8 has the right angle vertex at the center of a square of side 7, as shown. What is the area common to the triangle and the square?




A. $\frac{13\sqrt{2}}{4}$ B. $\frac{49}{4}$ C. 32

D. $\frac{8\pi}{3}$ E. $\sqrt{20}\pi$

27. If $51387^{453}$ is multiplied out, the units' digit in the final product is

A. 1 B. 3 C. 5 D. 9 E. 7

28. It is given that one root of $2x^2 + rx + s = 0$, where $r$ and $s$ are real numbers, is $5 - 3i$ $(i = \sqrt{-1})$. Find the value of $r + s$.

A. 48 B. -32 C. 32 D. -48 E. 10

29. In triangle PQR, PQ and PR are 4 cm. and 7 cm. respectively. The median $PM = 3\frac{1}{2}$ cm. Find QR in cm.

A. 8 B. $\sqrt{123}$ C. $\sqrt{80}$ D. 9 E. 11

30. What is the remainder when $10^{20}$ is divided by 98 ?

A. 2 B. 6 C. 32 D. 96 E. None of these

31. A rhombus is inscribed in triangle ABC in such a way that one of its vertices is A and two of its sides lie along $\overline{AB}$ and $\overline{AC}$. If AC = 6 cm., AB = 12 cm., and BC = 8 cm., the side of the rhombus, in cm., is:

A. 2 B. 3 C. $3\frac{1}{2}$ D. 4 E. 5

32. Line $\overline{CA}$ drawn on the floor is perpendicular to the intersection line $\overline{AB}$ of the (horizontal) floor and the (vertical) wall. Line $\overline{AD}$ drawn on the wall makes a 60° angle with edge $\overline{AB}$. Angle DAC is

A. 120° B. 90° C. 150° D. Between 120° and 150°
E. None of these

33. A communications network is to be set up having 10 stations. If each station is to have a direct line to three other stations, how many lines are necessary?

A. 15 B. 20 C. 21 D. 25 E. 30

34. If $x^{12} - 9$ is divided by $x + 2$, the remainder is:

A. -5005 B. 4087 C. -9/2 D. -10

E. None of these

35. If the triangle PQR is inscribed in a semicircle with $\overline{PQ}$ as a diameter, and if PQ denotes the length of the segment $\overline{PQ}$, then PR + QR must be

A. Equal to PQ B. Equal to PQ $\sqrt{2}$

C. Greater than or equal to PQ $\sqrt{2}$

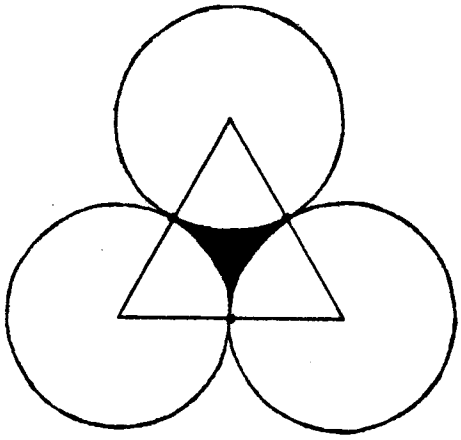
D. Less than or equal to PQ $\sqrt{2}$ E. None of these

36. For all values of $\theta$ the expression $2 \sin \theta - \sin 2\theta$ is equal to

A. $4 \sin \theta \sin^2 \frac{\theta}{2}$ B. $\csc 2\theta - 2 \csc \theta$ C. 0

D. $\sin \theta(2 - \sin \theta)$ E. None of these

37. Three circles of equal radius $r$ are mutually tangent as shown. What is the area of the shaded area in terms of $r$ ?



A. $\frac{r^2}{2}$ B. $\frac{2\sqrt{3}-\pi}{2} r^2$

C. $\frac{\sqrt{3}}{2} r^2$ D. $\frac{\sqrt{3}+2}{7} r^2$

E. None of these

38. An equilateral triangle is inscribed in a circle of radius 7 . What is the area of the triangle?

A. $\frac{7(1+\sqrt{3})}{4}$ B. $\frac{49(1-\sqrt{3})}{2}$ C. $\frac{147\sqrt{3}}{4}$

D. 29 E. None of these

39. If $a, b, c$ are positive, odd integers, the remainder when $a^2 + b^2 + c^2$ is divided by 4 is equal to

A. 1 B. 2 C. 3 D. 0 E. Not enough information

40. The smallest value for $x + \frac{4}{x}$ , where $x$ is a positive integer is

A. 1 B. 2 C. $\frac{13}{5}$ D. 4 E. $\frac{26}{5}$

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

DIRECTOR

A. C. Dempster
Eastern Michigan University

OFFICERS OF THE MICHIGAN SECTION

Chairperson

J. E. Adney, Jr.
Michigan State University

Vice-Chairpersons

D. G. Malm
Oakland University

D. Ross
Washtenaw Community College

Secretary-Treasurer

R. A. Chaffer
Central Michigan University

Governor

Y. Alavi
Western Michigan University

EXAMINATION COMMITTEE

Chairman

J. L. Ullman
University of Michigan

S. L. Rajnak
Kalamazoo College

E. A. Nordhaus
Michigan State University

T. Eisenberg
Northern Michigan University

ACKNOWLEDGMENTS

The following industries and professional organizations have provided generous financial support to this competition.

Burroughs Corporation
Michigan Bell Telephone
Michigan Council of Teachers of Mathematics
Kuhlman Corporation

The Michigan Association of Secondary School Principals has placed this competition on the Approved List of Michigan Contests and Activities.