20th ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

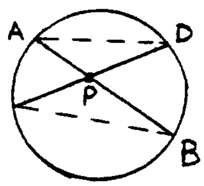
Sample Solutions for Part II

1. The total cost of 1 football, 3 tennis balls and 7 golf balls is \$14, while that of 1 football, 4 tennis balls and 10 golf balls is \$17. If one has \$20 to spend, is this sufficient to buy
   a) 3 footballs and 2 tennis balls?
   b) 2 footballs and 3 tennis balls?

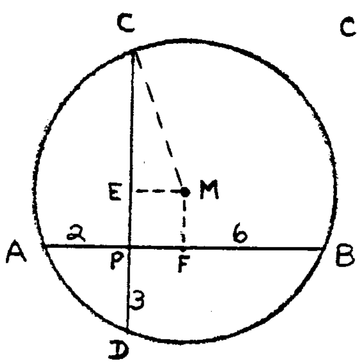
   Solution: Let f denote the cost of a football, t a tennis ball and g a golf ball. It is assumed that all are positive. Then $f + 3t + 7g = 14$ and $f + 4t + 10g = 17$. Solve for f and t in terms of g and obtain $f = 5 + 2g$, $t = 3 - 3g$. Then
   a) $3f + 2t = 15 + 6g + 6 - 6g = 21$, so the \$20 is not sufficient.
   b) $2f + 3t = 19 - 5g \lneqq 20$, so the \$20 is sufficient.

2. Let $\overline{AB}$ and $\overline{CD}$ be two chords in a circle intersecting at a point P (inside the circle).
   a) Prove that $AP \cdot PB = CP \cdot PD$.
   b) If $\overline{AB}$ is perpendicular to $\overline{CD}$ and the length of $\overline{AP}$ is 2, the length of $\overline{PB}$ is 6, and the length of $\overline{PD}$ is 3, find the radius of the circle.

   Solution:



   a) Construct $\overline{AD}$ and $\overline{BC}$. Then $m\angle BAD = m\angle BCD$ and $m\angle ABC = m\angle ADC$ as they intercept common arcs. Thus triangle APD is similar to triangle CPB. Hence $AP/PD = CP/PB$. So $AP \cdot PB = CP \cdot PD$.



   b) By Part a) $PC = 4$. Let M denote the center and "drop" perpendiculars to $\overline{CD}$ at E and to $\overline{AB}$ at F. Note that E is the midpoint of $\overline{CD}$ and F is the midpoint of $\overline{AB}$. Thus $ME = PF = 2$ and $CE = 7/2$. Using the right triangle CEM we obtain $r = MC = \frac{1}{2}\sqrt{65}$.

3. A polynomial $P(x)$ of degree greater than one has the remainder 2 when divided by $x-2$ and the remainder 3 when divided by $x-3$. Find the remainder when $P(x)$ is divided by x^2-5x+6.

   Solution: Using the Remainder Theorem we have $P(x) = (x-2)g(x) + 2$ so $P(2) = 2$ and $P(x) = (x-3)k(x) + 3$ so $P(3) = 3$. Substitute $x=3$ in the top equation to get $3 = g(3) + 2$. Thus $g(3) = 1$ so $g(x) = (x-3)h(x) + 1$. Then $P(x) = (x-2)((x-3)h(x) + 1) + 2 = (x-2)(x-3)h(x) + x$. Therefore the remainder is x.

   Alternate Solution: Since the remainder is of degree at most 1 it has the form $ax + b$. That is, $P(x) = (x-2)(x-3)f(x) + (ax + b)$. Then $2 = P(2) = 2a + b$ and $3 = P(3) = 3a + b$. Solving these two equations for a and b yields $a = 1$, and $b = 0$.

4. Let $x_1 = 2$ and $x_{n+1} = x_n + (3n+2)$ for all n greater than or equal to one.

a) Find a formula expressing x_n as a function of n.

b) Prove your result.

Solution:

a) Note that $x_n = x_{n-1} + (3(n-1) + 2) = x_{n-1} + 3n - 1$. Thus the sequence $x_1, x_2, x_3, \ldots$ is $2, 2+(3\cdot 2 - 1), 2+(3\cdot 2 - 1) + (3\cdot 3 - 1)$, etc. As $2 = 3\cdot 1 - 1$, we obtain

$$x_n = \sum_{j=1}^{n} (3j-1) = 3(\sum_{j=1}^{n} j) - n = 3(\frac{n(n+1)}{2}) - n. \quad \text{Thus } x_n = \frac{3n^2 + n}{2}$$

b) Proof of result (by induction). Let P_n be the statement: " $x_n = (3n^2 + n)/2$ ". Since $(3(1)^2 + 1)/2 = 2$, $P(1)$ is true. Now assume P_k is true. Then

$$x_{k+1} = x_k + (3k+2) = (3k^2 + k)/2 + (3k+2)$$
$$= (3k^2 + 7k + 4)/2 = (3(k+1)^2 + (k+1))/2.$$

Hence $P(k)$ true does imply $P(k+1)$ true. Done.

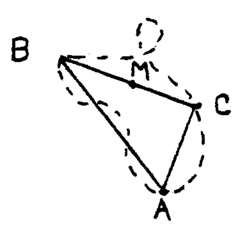
5. The point M is the midpoint of side $\overline{BC}$ of a triangle ABC.

a) Prove that $AM \leq \frac{1}{2}AB + \frac{1}{2}AC$.

b) A fly takes off from a certain point and flies a total distance of 4 meters, returning to the starting point. Explain why the fly never gets outside of some sphere with a radius of one meter.

Solution:

a) Let D be the midpoint of $\overline{AB}$, E the midpoint of $\overline{AC}$. Then $\overline{EM} \parallel \overline{AB}$, $\overline{DM} \parallel \overline{AC}$, $EM = \frac{1}{2}AB$ and $AE = \frac{1}{2}AC$. In triangle AME, $AM \leq EM + AE = \frac{1}{2}AB + \frac{1}{2}AC$.



b) Let B denote the starting point of the journey and C the 2 meter mark. Thus the length of the line $\overline{BC}$ is at most 2. Let M denote the midpoint of $\overline{BC}$. Let A denote a point on the path which is a maximum distance from M. Then $BA + AC \leq 2$, the length of the path connecting B and C through A. Using triangle ABC and part a), $MA \leq \frac{1}{2}(AB + AC) \leq \frac{1}{2}(2) = 1$. Since A is located a maximum distance from M, there are no points on the journey located outside a sphere of radius 1 meter, centered at M.