1. The total cost of 1 football, 3 tennis balls and 7 golf balls is $14, while that of
1 football, 4 tennis balls and 10 golf balls is $17. If one has $20 to spend, is this
sufficient to buy
a) 3 footballs and 2 tennis balls?
b) 2 footballs and 3 tennis balls?

Solution: Let \( f \) denote the cost of a football, \( t \) a tennis ball and \( g \) a golf ball.
It is assumed that all are positive. Then \( f + 3t + 7g = 14 \) and \( f + 4t + 10g = 17 \).
Solve for \( f \) and \( t \) in terms of \( g \) and obtain \( f = 5 + 2g \), \( t = 3 - 3g \). Then
a) \( 3f + 2t = 15 + 6g + 6 - 6g = 21 \), so the $20 is not sufficient.
b) \( 2f + 3t = 19 - 5g \leq 20 \), so the $20 is sufficient.

2. Let \( \overline{AB} \) and \( \overline{CD} \) be two chords in a circle intersecting at a point \( P \) (inside the
circle).

a) Prove that \( \overline{AP} \cdot \overline{PB} = \overline{CP} \cdot \overline{PD} \).
b) If \( \overline{AB} \) is perpendicular to \( \overline{CD} \) and the length of \( \overline{AP} \) is 2, the length of
\( \overline{PB} \) is 6, and the length of \( \overline{PD} \) is 3, find the radius of the circle.

Solution:

a) Construct \( \overline{AD} \) and \( \overline{BC} \). Then \( m \angle BAD = m \angle BCD \) and \( m \angle ABC = m \angle ADC \)
as they intercept common arcs. Thus triangle \( \triangle APD \) is similar to triangle \( \triangle CPB \).
Hence \( \frac{AP}{PD} = \frac{CP}{PB} \). So \( AP \cdot PB = CP \cdot PD \).

b) By Part a) \( PC = 4 \). Let \( M \) denote the center and "drop" perpendiculars to \( \overline{CD} \) at \( E \) and to \( \overline{AB} \) at \( F \).
Note that \( E \) is the midpoint of \( \overline{CD} \) and \( F \) is the midpoint of \( \overline{AB} \). Thus \( ME = PF = 2 \) and \( CE = 7/2 \).
Using the right triangle \( \triangle CEM \) we obtain \( r = MC = \frac{1}{2} \sqrt{65} \).

3. A polynomial \( P(x) \) of degree greater than one has the remainder 2 when divided
by \( x - 2 \) and the remainder 3 when divided by \( x - 3 \). Find the remainder when \( P(x) \)
is divided by \( x^2 - 5x + 6 \).

Solution: Using the Remainder Theorem we have \( P(x) = (x-2)g(x) + 2 \) so \( P(2) = 2 \)
and \( P(x) = (x-3)h(x) + 3 \) so \( P(3) = 3 \). Substitute \( x = 3 \) in the top equation to get
\( 3 = g(3) + 2 \). Thus \( g(3) = 1 \) so \( g(x) = (x-3)h(x) + 1 \). Then
\( P(x) = (x-2)((x-3)h(x) + 1) + 2 = (x-2)(x-3)h(x) + x \). Therefore the remainder is \( x \).

Alternate Solution: Since the remainder is of degree at most 1 it has the form \( ax + b \).
That is, \( P(x) = (x-2)(x-3)f(x) + (ax + b) \). Then \( 2 = P(2) = 2a + b \) and
\( 3 = P(3) = 3a + b \). Solving these two equations for \( a \) and \( b \) yields \( a = 1 \), and
\( b = 0 \).
4. Let $x_1 = 2$ and $x_{n+1} = x_n + (3n+2)$ for all $n$ greater than or equal to one.
   a) Find a formula expressing $x_n$ as a function of $n$.
   b) Prove your result.

Solution:
   a) Note that $x_n = x_{n-1} + (3(n-1) + 2) = x_{n-1} + 3n - 1$. Thus the sequence $x_1, x_2, x_3, \ldots$
   is 2, 2+(3·2 - 1), 2+(3·2 - 1) + (3·3 - 1), etc. As $2 = 3·1 - 1$, we obtain
   
   $$x_n = \sum_{j=1}^{n} (3j-1) = 3\sum_{j=1}^{n} j - n = 3\left(\frac{n(n+1)}{2}\right) - n.$$  
   Thus $x_n = \frac{3n^2 + n}{2}$

   b) Proof of result (by induction). Let $P_n$ be the statement: "$x_n = (3n^2 + n)/2$".
   Since $(3(1)^2 + 1)/2 = 2$, $P(1)$ is true. Now assume $P_k$ is true. Then
   $x_{k+1} = x_k + (3k+2) = \left(\frac{3k^2 + k}{2}\right) + (3k+2)$
   $= \left(3k^2 + 7k + 4\right)/2 = \left(3(k+1)^2 + (k+1)\right)/2$.
   Hence $P(k)$ true does imply $P(k+1)$ true. Done.

5. The point $M$ is the midpoint of side $BC$ of a triangle $ABC$.
   a) Prove that $AM \leq \frac{1}{2}AB + \frac{1}{3}AC$.
   b) A fly takes off from a certain point and flies a total distance of 4 meters,
      returning to the starting point. Explain why the fly never gets outside of
      some sphere with a radius of one meter.

Solution:
   a) Let $D$ be the midpoint of $AB$, $E$ the midpoint of $AC$. Then $EM \parallel AB$,
   $DM \parallel AC$, $EM = \frac{1}{2}AB$ and $AE = \frac{1}{2}AC$. In triangle $AME$, $AM \leq EM + AE = \frac{1}{2}AB + \frac{1}{2}AC$.

   b) Let $B$ denote the starting point of the journey
   and $C$ the 2 meter mark. Thus the length of the
   line $BC$ is at most 2. Let $M$ denote the midpoint
   of $BC$. Let $A$ denote a point on the path which
   is a maximum distance from $M$. Then $BA + AC \leq 2$,
   the length of the path connecting $B$ and $C$ through
   $A$. Using triangle $ABC$ and part a),
   $MA \leq \frac{1}{2}(AB + AC) \leq \frac{1}{2}(2) = 1$. Since $A$ is located
   a maximum distance from $M$, there are no points
   on the journey located outside a sphere of radius
   1 meter, centered at $M$. 

\[ \text{Diagram of triangle} \]