TWENTIETH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America with the assistance of Michigan Colleges and Universities, Professional Organizations, and Industries.

PART 1

October 20, 1976

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Check to insure that your six-digit student number has been recorded correctly. Do not make calculations on the answer sheet.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. The first 20 problems of this examination are intended to sample many of the topics in the secondary mathematics curriculum. You may be unfamiliar with some of these topics and quite possibly will find a number of problems which are easier for you distributed throughout the last twenty items. Usually a score of about 20 or more will allow you to become a finalist and write the second exam.

5. In each of the questions five different possible responses are proposed. In some cases the fifth alternative is listed "E, none of these". In such cases if you believe none of the first four alternatives to be correct, mark E.

6. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer or are able to eliminate some of the alternatives as impossible.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. The use of books, tables, slide rules or electronic calculators is prohibited. If you have questions concerning the instructions ask them now.

20th ANNUAL MICHIGAN MATHEMATICS

PRIZE COMPETITION

1. The fraction $\frac{mq - p + mp - q}{p + q}$ when simplified is equal to

   A. $m - p - q$ B. $m(p + q)$ C. $m - 1$ D. $m(p - q)$ E. $-p - q$

2. When $x = 9$ the value of $5x^0 + \left(\frac{3}{x}\right)^{-2} + \sqrt{x^3}$ is

   A. 19 B. 23 C. 37 D. 41 E. None of these

3. If $9x^2 + 30x + k^2$ is a perfect square, then one value of k is

   A. 6 B. 5 C. 9 D. 16 E. 28

4. The solution of the equation $\frac{2x - 3}{2} + \frac{x - 4}{3} = 0$ is

   A. $2\frac{1}{8}$ B. $2\frac{6}{4}$ C. 2 D. $2\frac{1}{2}$ E. $2\frac{1}{3}$

5. The sum $(x - y)^{-1} + (x + y)^{-1}$ equals

   A. $(x^2 - y^2)(2x)^{-1}$ B. $(x^2 - y^2)^{-1}$ C. $(2x)^{-1}$

   D. $2x(x^2 - y^2)^{-1}$ E. $2x(y^2 - x^2)^{-1}$

6. Which of the following is the smallest?

   A. $\frac{1}{700}$ B. $2^{1/10}$ C. 10^{-3} D. .002 E. 2^{-10}

7. A circle is circumscribed about an isosceles right triangle whose short sides have length 1. What is the area of the circle?

   A. π B. $\frac{\pi}{2}$ C. 2π D. $2\sqrt{2}\pi$ E. None of these

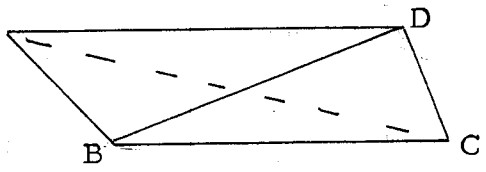
8. If $2x + iy - 3y + 2i = 0$, where $i^2 = -1$ and x and y are real numbers, then the sum of x and y equals

A. -5 B. $\frac{-2}{3}$ C. 0 D. 1 E. -1

9. S is a set of 10 elements. P is a subset of S with 7 elements and Q is a subset of S with 5 elements. Which of the following is always true?

A. $P \cap Q = \phi$ B. $P \cup Q = S$ C. $P \cap Q \neq \phi$ D. $Q \subset P$

E. None of these

10. 

The triangle BDC is right-angled at D and $\overline{AD}$ is parallel to $\overline{BC}$. If BC = 10 cm. and DC = 6 cm. then the area of the triangle ABC in square centimeters is

A. 24 B. 30 C. 20 D. 36 E. 18

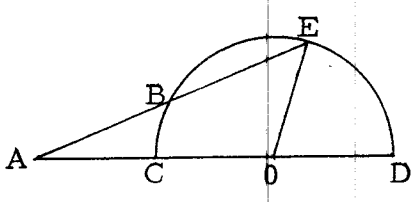
11. If the cost of producing one shoe is x, then the cost of producing y pairs of shoes is

A. $\frac{2x}{y}$ B. $2xy$ C. $\frac{y}{2x}$ D. $y - 2x$ E. None of these

12. If $x = -1$ is a root of the equation $a(x-2)^4 + b(x-2)^2 + c = 0$, then another root is

A. -3 B. 0 C. 1 D. 3 E. 5

13. A semicircle with diameter $\overline{CD}$ and center 0 is cut by a secant $\overline{ABE}$, as shown. If the measure of $\angle EOD$ is $75°$, and $AB = CO$, then the measure of $\angle BAC$ is



A. $15°$ B. $20°$ C. $28°$ D. $30°$ E. $25°$

14. If $3^x = 4(3^{-x})$, then x equals

A. $\frac{\log 2}{\log 3}$ B. $\frac{\log 4}{\log 3}$ C. $\log\left(\frac{4}{9}\right)$ D. 4 E. $\frac{2 \log 2}{3 \log 3}$

15. If $x - y = 4$ and $xy = 2$, then $|x + y|$ equals

A. 0 B. 6 C. $2\sqrt{6}$ D. 8 E. $\sqrt{6}$

16. A spherical balloon is inflated,still remaining spherical so that its volume is eight times the original. If the radius of the original was 5 cm., then the new radius in centimeters is

A. 20 B. 8 C. 12 D. 15 E. 10

17. A card is selected at random from a standard deck of 52 playing cards. The probability that it is either a club or an ace (or both) is

A. $\frac{1}{4}$ B. $\frac{3}{13}$ C. $\frac{4}{13}$ D. $\frac{17}{52}$ E. $\frac{1}{3}$

18. If $\log_{10} x = 3\log_{10} 2 - 2$, then x equals

A. .6 B. .223 C. $10^{2/3}$ D. 10^6 E. .08

19. The expression $(\sin\theta + \cos\theta)^2 - 1$ is equal to

A. $\cos 2\theta$ B. $\sin 2\theta$ C. $\tan 2\theta$ D. $\sec 2\theta$ E. 0

20. If $2 - 3x + x^{19}$ is divided by $x + 1$, the remainder is

A. -2 B. 0 C. 6 D. 4 E. None of these

21. If $\left|\frac{x-1}{x+3}\right| < 1$, then the solution set is

A. x is every real number $x \neq 3$ B. $-3 < x < 1$

C. all $x < -1$, $x \neq -3$ D. all $x > -1$

E. $-1 < x < 1$

22. A function $f(x)$ is defined on the set of integers by the formula $f(x) = a - 3x$. Given that $f(1) = 1$, then $f(2)$ is

A. -2 B. 4 C. 3 D. -3 E. 2

23. In the arithmetic series $5 + 16 + 27 + 38 + \cdots$ the sum of the first n terms is

A. $\frac{n}{2}[22 + (n-1)5]$ B. $\frac{n}{2}(11n - 1)$ C. $\frac{n}{2}(11n + 1)$

D. $\frac{n}{2}(11n - 5)$ E. $\frac{n}{2}(11n - 2)$

24. The sum of the roots of $2x^2 + 7x - 22 = 0$ is

A. $\frac{7}{2}$ B. $\frac{-9}{2}$ C. -5 D. $\frac{15}{2}$ E. $\frac{-7}{2}$

25. The repeating decimal $1.4363636\ldots$ where the digits 3 and 6 continue to repeat represents the same rational number as

A. 1.44 B. $\frac{359}{250}$ C. $\frac{79}{55}$ D. $\frac{3591}{2500}$ E. None of these

26. The largest value of n for which 20^n divides the product of the first 20 positive integers is

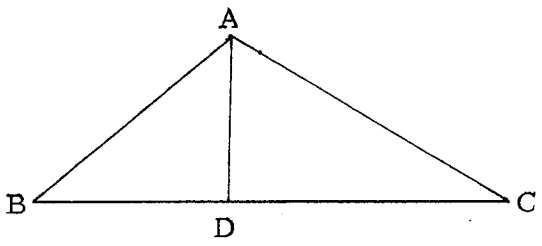
A. 1 B. 4 C. 6 D. 10 E. 20

27. What is the following sum:

$\cos 0^\circ + \cos 180^\circ + \cos(2 \times 180^\circ) + \cos(3 \times 180^\circ) + \cdots + \cos(100 \times 180^\circ)$?

A. 0 B. 100 C. 1 D. 101 E. 180

28.



In the triangle ABC, $\overline{AD}$ is perpendicular to $\overline{BC}$ and AB = 5 cm. If the measure of $\angle$ ABC is 40° and the measure of $\angle$ ACB is 30°, the length of $\overline{DC}$ in centimeters is

A. $5 \sin 40^\circ \tan 30^\circ$ B. $5 \sin 40^\circ \cos 30^\circ$ C. $5 \cos 40^\circ \tan 30^\circ$

D. $5 \cos 40^\circ \cos 60^\circ$ E. $5 \sin 40^\circ \tan 60^\circ$

29. For an arbitrary triangle the center of the inscribed circle is the point of intersection of

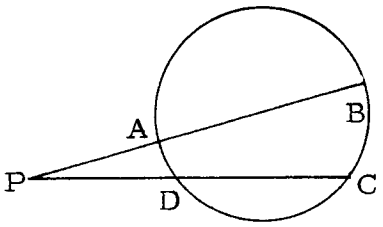
A. the medians of the triangle.

B. the perpendicular bisectors of the sides of the triangle.

C. the bisectors of the internal angles of the triangle.

D. the altitudes of the triangle.

E. none of the above.

30. 

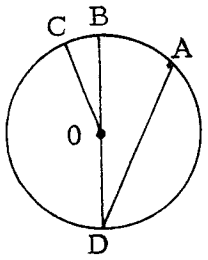
In the following diagram $\overline{PA}$ is 12 cm. shorter than $\overline{PD}$. Given that the chord AB = 156 cm. and the chord CD = 96 cm. then PA equals

A. 36 cm. B. 24 cm. C. 48 cm. D. 30 cm. E. 16 cm.

31. A line passing through the point (1,1) and perpendicular to the line through the points (0,3) and (2,11) has the equation

A. $x - 3y + 2 = 0$ B. $x + 3y - 4 = 0$ C. $x - 2y + 1 = 0$

D. $x - 4y + 3 = 0$ E. $x + 4y - 5 = 0$

32. 

Point 0 is the center of a circle of radius 9. If the measure of $\angle$ ADB is 22° and the measure of $\angle$ BOC is 16°, then the arc ADC has length

A. 8π B. 14π C. $9\pi \cos 38^\circ$ D. 15π E. None of these

33. Given a triangle whose sides have lengths a, b, and c, under what conditions is it possible to have $c^2 > a^2 + b^2$?

A. It is a right triangle.

B. The triangle is equilateral.

C. The angle opposite the side of length c is acute.

D. The angle opposite the side of length c is obtuse.

E. It can never be true.

34. How many distinct 6 digit numbers can be formed by using the digits 2, 2, 2, 4, 4, 4 ?

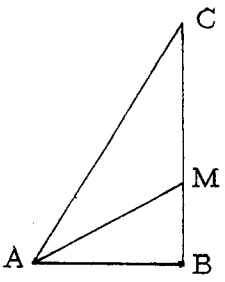
A. 20 B. 6 C. 9 D. 36 E. 64

35. If two men working at the same rate can paint a car in 12 hrs. then at that rate the number of hours for three men to do the job is

A. 9 B. 8 C. 6 D. $\frac{20}{3}$ E. 7

36. If x is real, then the minimum value of $x^2 + 8x$ is

A. -8 B. -16 C. 0 D. 9 E. 48

37. 

In the triangle, the side AC = 20 cm. AB = 10 cm. and the angle at B is 90°. If $\overline{AM}$ bisects the angle at A, then BM equals

A. 30 cm. B. $10\sqrt{3}$ cm. C. $\frac{10\sqrt{3}}{2}$ cm. D. $\frac{10\sqrt{3}}{3}$ cm.

E. $\frac{20}{\sqrt{3}}$ cm.

38. A man drives from one city to another and then returns on the same route. If his average speed going is r kilometers per hour and his average speed for the entire trip is s kilometers per hour, then his average speed returning, in kilometers per hour, is

A. $\frac{r+s}{2}$ B. $2s - r$ C. $\frac{rs}{2r-s}$ D. $\frac{2rs}{r+s}$

E. None of these

39. A polynomial $P(x)$ has integer coefficients. If $1, \sqrt{3}, i$ are roots of $P(x)$, and $P(x)$ has lowest possible degree, which of the following is <u>not</u> a factor of $P(x)$?

A. $x - \sqrt{3}$ B. $x^2 + 1$ C. $x^2 - 3$ D. $x + 1$ E. $x + i$

40. The average value of all the roots of the equation $2x^{10} - 180x^9 + 5x + 7 = 0$ is

A. 9 B. 18 C. $\frac{3}{20}$ D. imaginary E. irrational

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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