19th ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Sample Solutions for Part II

The sketch of one solution for each of the Part II problems is given below. Solutions do not include possible generalizations.

1. a) Given four points in the plane, no three of which lie on the same line, each subset of three points determines the vertices of a triangle. Can all these triangles have equal areas? If so, give an example of four points (in the plane) with this property, and then describe all arrangements of four points (in the plane) which permit this. If no such arrangement exists, prove this.
b) Repeat part a) with "five" replacing "four" throughout.

Solution: a) Four Points. The desired arrangement is to have the four points represent the vertices of a parallelogram. Pick A and B so that the other two points are on the same side of the line $\overleftrightarrow{AB}$. The other points, C and D, have to be the same distance from $\overleftrightarrow{AB}$ in order that the two triangles having $\overline{AB}$ as a base will have the same height and thus equal areas. Hence $\overline{AB}$ is parallel to $\overline{CD}$. If the length CD is different from AB, then triangles ABC and CDA will have different areas (same height, different base). Thus ABCD is a parallelogram.

b) Five Points. There is no acceptable solution now. Pick two points A and B such that the other three points are on the same side of the line $\overline{AB}$. Then, to have equal areas in the resulting triangles, the other three points must be at the same distance from $\overline{AB}$. However, this forces them to be collinear.

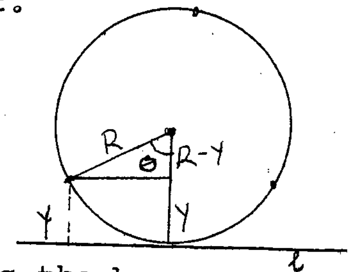
2. Three people at the base of a long stairway begin a race up the stairs. Person A leaps five steps with each stride (landing on steps 5, 10, 15, etc.). Person B leaps a little more slowly but covers six steps with each stride. Person C leaps seven steps with each stride. A picture taken near the end of the race shows all three landing simultaneously, with Person A twenty-one steps from the top, Person B seven steps from the top, and Person C one step from the the top. How many steps are there in the stairway? If you can find more than one answer, do so. Justify your answer.

Solution: Let N be the number of steps. Then 5 divides N-21, so 5 divides N-1-20 which implies 5 divides N-1. Similarly 6 divides N-7 and also N-1 and 7 divides N-1. Since 5, 6, and 7 divide N-1 and these are pairwise relatively prime, their product 210 divides N-1 as well. Hence N-1 = 210k, so the number of steps N = 210k + 1 where k is a positive integer.

3. Let S denote the sum of an infinite geometric series. Suppose the sum of the squares of the terms is 2S, and that of the cubes is 64S/13. Find the first three terms of the original series.

Solution: We are given (1) $S = \sum_{n=0}^{\infty} ar^n = a/(1-r)$,
(2) $2S = \sum_{n=0}^{\infty} (ar^n)^2 = \sum_{n=0}^{\infty} a^2 (r^2)^n = a^2/(1-r^2)$,
and (3) $64S/13 = \sum_{n=0}^{\infty} (ar^n)^3 = \sum_{n=0}^{\infty} a^3 (r^3)^n = a^3/(1-r^3)$. Combining (1) and (2) yields $2(a/(1-r)) = a^2/(1-r^2)$ and so, (4), $a = 2(1+r)$. Combining (1) and (3) gives (5) $64/13 = a^2/(1+r+r^2)$. Working with (4) and (5) we obtain $64/13 = 4(1+r)^2/(1+r+r^2)$ so $16(1+r+r^2) = 13(r^2+2r+1)$. Simplification yields $3r^2-10r+3 = 0$ which has solutions $r = 3$ and $r = 1/3$. However, $r = 3$ gives a divergent series. Thus $r = 1/3$. Using (4), $a = 8/3$. The first three terms are a, ar, and ar^2. That is, 8/3, 8/9 and 8/27.

4. A, B, and C are three equally spaced points on a circular hoop. Prove that as the hoop rolls along the horizontal line ℓ, the sum of the distances of the points A, B, and C above line ℓ is constant.



Solution: Let R be the radius of the hoop and θ the angle from the vertical as indicated. Then, $\cos\theta = (R-y)/R$ so $y = R - R\cos\theta$. Thus the sum of the three distances is

$$S = (R - R\cos\theta) + (R - R\cos(\theta+120^\circ)) + (R - R\cos(\theta+240^\circ))$$
$$= 3R - R(\cos\theta + \cos(\theta+120^\circ) + \cos(\theta+240^\circ)).$$

Now apply $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ twice, evaluating the known factors, and obtain $S = 3R - 0 = 3R$.

5. A set of n numbers $x_1, x_2, x_3, \ldots, x_n$ (where $n>1$) has the property that the k^{th} number (that is, x_k) is removed from the set, the remaining $(n-1)$ numbers have a sum equal to k (the subscript of x_k), and this is true for each $k = 1,2,3,\ldots,n$.

a) Solve for these n numbers.
b) Find whether at least one of these n numbers can be an integer.

Solution: Let S denote the sum $x_1 + x_2 + x_3 + \ldots + x_n$. Then (1) $S - x_1 = 1$ and (2) $S - x_k = k$. Subtract (2) from (1) to get $x_k - x_1 = 1 - k$. Thus, (3) $x_k = x_1 + 1 - k$. This allows substitution into (1) to obtain:

$$1 = x_2 + x_3 + \ldots + x_n$$
$$= (x_1+1-2) + (x_1+1-3) + \ldots + (x_1+1-n)$$
$$= (x_1-1) + (x_1-2) + \ldots + (x_1-(n-1)).$$

Thus $1 = (n-1)x_1 - (1+2+3+\ldots+(n-1)) = (n-1)x_1 - n(n-1)/2$.

Therefore, $x_1 = 1/(n-1) + n/2$. From (3), $x_k = 1/(n-1) + n/2 + 1 - k$.

For $n>1$, x_k can be an integer only in the cases $n = 2$, $(k = 1$ or $2)$ or $n = 3$, $(k = 1, 2,$ or $3)$.