18th Annual Michigan Mathematics Prize Competition

Sample Solutions for Part II

The sketch of one solution for each of the Part II problems is given below. Solutions do not include possible generalizations.

1. Let S be the sum of the 99 terms:

$$(\sqrt{1}+\sqrt{2})^{-1},\ (\sqrt{2}+\sqrt{3})^{-1},\ (\sqrt{3}+\sqrt{4})^{-1},\ \ldots,\ (\sqrt{99}+\sqrt{100})^{-1}.$$

Prove that S is an integer.

Solution: $S = (\sqrt{1}+\sqrt{2})^{-1} + (\sqrt{2}+\sqrt{3})^{-1} + \ldots + (\sqrt{99}+\sqrt{100})^{-1}$

$$= \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \ldots + \frac{1}{\sqrt{99}+\sqrt{100}}$$

Multiplying numerator and denominator of each fraction by the difference of the radicals appearing yields:

$$S = (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \ldots + (\sqrt{100}-\sqrt{99})$$

Noting that $\sqrt{n}$ and $-\sqrt{n}$ occur for all $n = 2, 3, \ldots, 99$ permits the simplification:

$S = -\sqrt{1} + \sqrt{100} = -1 + 10 = 9$ which is an integer.

2. Determine all pairs of positive integers x and y for which $N = x^4 + 4y^4$ is a prime.

Solution: The question of primeness suggests that we try to represent N in factored form. Complete the square:

$$N + 4x^2y^2 = x^4 + 4x^2y^2 + 4y^4 = (x^2 + 2y^2)^2$$

$$\therefore N = (x^2 + 2y^2)^2 - 4x^2y^2$$

$$= (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$$

Thus N will be prime if and only if

$$N = x^2 + 2y^2 + 2xy \text{ and } 1 = x^2 + 2y^2 - 2xy$$

The second equation may be rewritten as

$$(x-y)^2 + y^2 = 1 \Rightarrow x = y = 1 \text{ and } N = 5.$$

3. Let w, x, y, z be arbitrary positive real numbers. Prove each inequality:

a) Prove $xy \le \left(\frac{x+y}{2}\right)^2$

$(x-y)^2 \ge 0 \Rightarrow x^2 - 2xy + y^2 \ge 0$ also $x^2 + 2xy + y^2 = (x+y)^2$ multiply both sides of the equality by (-1) and adding to the inequality yields: $-4xy \ge -(x+y)^2$ or $xy \le \left(\frac{x+y}{2}\right)^2$

b) Prove $wxyz \le \left(\frac{w+x+y+z}{4}\right)^4$

From part "a" we know: $wx \le \left(\frac{w + x}{2}\right)^2$ and $yz \le \left(\frac{y + z}{2}\right)^2$

all terms positive permit the multiplication of these inequalities to yield:

$$wxyz \le \left[\left(\frac{w + x}{2}\right)\left(\frac{y + z}{2}\right)\right]^2 \le \left[\left(\frac{\frac{w + x}{2} + \frac{y + z}{2}}{2}\right)^2\right]^2$$

where the second inequality is obtained by an additional application of part "a". The last quantity is precisely the one desired.

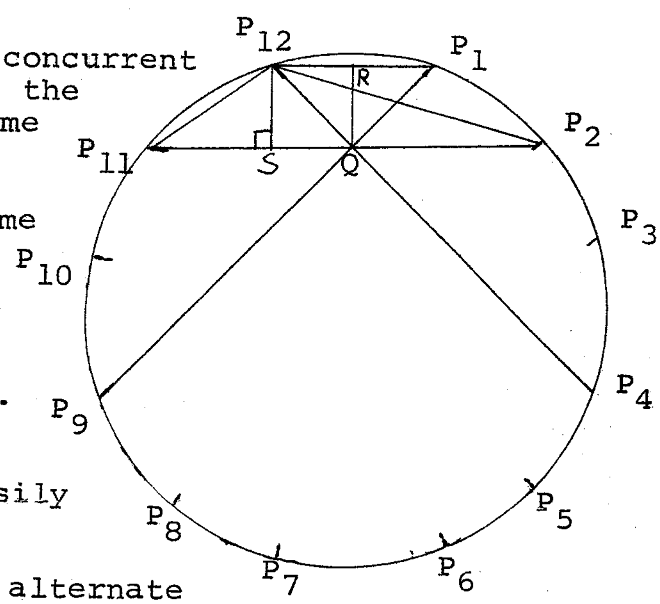
c) Prove $xyz \le \left(\frac{x + y + z}{3}\right)^3$

Use part "b" letting $w = \frac{x + y + z}{3}$ which yields:

$$wxyz \le \left[\frac{\frac{x + y + z}{3} + x + y + z}{4}\right]^4 = \left(\frac{x + y + z}{3}\right)^4 = w^4$$

$wxyz \le w^4 \Leftrightarrow xyz \le w^3 = \left(\frac{x + y + z}{3}\right)^3$ where the division by w is permitted since $w > 0$.

4. Solution: To prove the three lines are concurrent we shall prove that the point Q which is the intersection of $\overline{P_1P_9}$ and $\overline{P_4P_{12}}$ is the same distance from $\overline{P_1P_{12}}$ as the line $\overline{P_2P_{11}}$ which is parallel to $\overline{P_1P_{12}}$ and on the same side as Q.

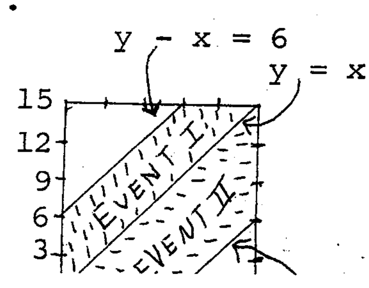


Without loss of generality, let $P_1P_{12} = 2$ units and R the point where the bisector of $\angle P_1QP_{12}$ intersects $\overline{P_1P_{12}}$. Then since ΔP_1QP_{12} is isosceles with 45° base angles (inscribed angles), it is easily shown that $\overline{QR} \perp \overline{P_1P_{12}}$ and $QR = 1$.

Now $\overline{P_1P_{12}}$ is parallel to $\overline{P_2P_{11}}$ since the alternate interior angles $\angle P_2P_{12}P_1$ and $\angle P_{11}P_2P_{12}$ are congruent. Let $\overline{P_{12}S}$ be perpendicular to $\overline{P_{11}P_2}$ then $\Delta P_{12}P_{11}S$ is a 30°-60°-90° triangle with hypotenuse of 2 units. Then $P_{12}S$ (the distance from $\overline{P_{12}P_1}$ to $\overline{P_2P_{11}}$) is also 1 unit.

5. Two very busy men, A and B, who wish to confer, agree to appear at a designated place on a certain day, but no earlier than noon and no later than 12:15 p.m. If necessary, A will wait 6 minutes for B to arrive, while B will wait 9 minutes for A to arrive but neither can stay past 12:15 p.m. Express as a percent their chance of meeting.

Solution: Let A arrive x minutes past noon and B at y minutes past noon. Then $0 \le x \le 15$ and $0 \le y \le 15$. If A arrives first or both arrive simultaneously, then $y \ge x$ and $0 \le y - x \le 6$ if the men are to meet (Event I). Similarly if B arrives first then $x > y$ and $0 < x - y \le 9$ (Event II). The probability of each event is the ratio of the area of the ap-



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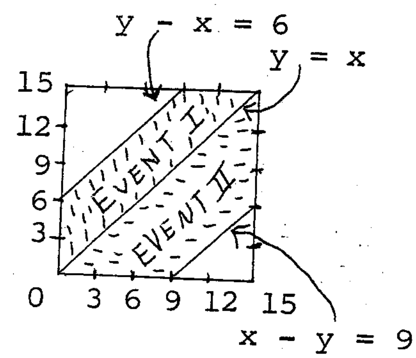

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$$\frac{225 - \frac{1}{2}(81) - \frac{1}{2}(36)}{225} = 74\%$$