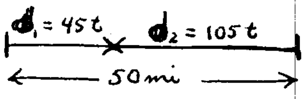
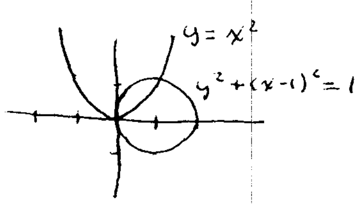
18th ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Part I Answers

No.	Ans.	Comments. (Many problems can be solved in several ways.)
1	C	It would take 1 man 36 hours and two men one-half as long.
2.	B	$1+26,\ 2+25,\ \ldots,\ 13+14$ Beyond one-half of 27 repeats combinations
3.	A	$5! = 5\times4\times3\times2\times1$ since 5 choices for the ~~book~~ first, 4 choices for the second,
4.	E	
5.	C	~~$x^4-4x^2+3=0 \Leftrightarrow (x^2-1)(x^2-3)=0 \Leftrightarrow (x+1)(x-1)(x+\sqrt{3})(x-\sqrt{3})=0$ Factor.~~
6.	A	The first sentence implies that the value is dependent only upon the weight. Thus the value of a ton is twice the value of half a ton.
7.	C	~~log~~ $10^{\log 3+\log 4+\log 5} = 10^{\log 3}\times10^{\log 4}\times10^{\log 5} = 3\times4\times5 = 60$
8.	B	1st hour, 2 cells; 2nd hour, 4 cells; ... nth hour, 2^n cells. Solve: $2^n \geq 1000$ and choose the least value of n.
9.	A.	Each of the $2n$ houses is connected to the $2n-1$ other houses yielding $2n(2n-1)$ wires. But this counts each wire twice thus divide by 2.
10.	D	The number of ~~permutations~~ permutations of the 4 remaining letters is 4!
11.	B	$1/(2+x) = 1/2 + 1/x \Leftrightarrow x^2+2x+4=0$ for $x \notin \{-2,0\}$ Apply Quadratic Formula or complete the square.
12.	C	 $d_1+d_2=50 \Leftrightarrow 45t+105t=50 \Leftrightarrow t = 1/3$ hr.
13.	A.	Let h = no. of hits and $(100-h)$ the number of misses $\therefore\ 25h-10(100-h) = -90$, $h=26$ and $100-h=74$.
14.	D	
15	A	$\Delta XYC \sim \Delta ABC \therefore r/p = CX/CA$; $\Delta AYX \sim \Delta ADC \therefore r/q = AX/CA$; Add equations and simplify.
16	E	~~More than one possible solution~~ Insufficient information for a unique solution
17	A	Grouping consecutive pairs of terms, each having value 1, yields $50(1)-101 = -51$
18	C	$3^{x+1}-3^{x-1}=24 \Leftrightarrow 3^x(3-3^{-1})=24 \Leftrightarrow 8/3\,(3^x)=24$, $x=2$, $y=26$
19	E	None satisfies each of the reflexive, symmetric, and transitive properties
20	C	

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21 E $4x^2 - 12x > 0 \Leftrightarrow x(x-3) > 0$ ~~which is~~ satisfied by all values for which both factors are positive or both negative. $\therefore x < 0$ or $x > 3$

22. E $|x| > \frac{1}{x} \Leftrightarrow [(x > 0 \text{ and } x > 1/x) \text{ or } (x < 0 \text{ and } -x > 1/x)] \Leftrightarrow (x > 1 \text{ or } x < 0)$

23 B

24 B $10^{30} = (10^3)^{10} = (998+2)^{10}$ which has the same remainder as 2^{10} since each ~~and~~ term of the expansion contains a factor of 998. $2^{10} = 1024$, $r = 26$

25 A Let n_i represent the number of cash prizes won by the i^{th} contestant and (n_1, n_2, n_3) an outcome of the contest. The number of different outcomes of the contest is the number of distinct permutations for each of $(0,0,4)$, $(0,1,3)$, $(0,2,2)$ and $(1,1,2)$ which is 3, 6, 3, and 3 respectively. The sum is 15.

26 B Solve the three equations simultaneously yielding $b = 1/2$. Thus $x = -1/2$, $y = -1/2$

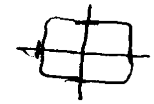
28 A $\overrightarrow{PL}$ is perpendicular to the half plane containing Q and thus is perpendicular to every line ~~in that half plane containing point P~~ which contains point P and lies in that plane.

27 D 10^{10} is the largest of the first three and $(2^{10})^5 = (2^5)^{10} > 10^{10}$ so that the largest is one of the last two. Consider $(2^{10})^5/(5^{10})^2 = (2^5)^{10}/(5^2)^{10} = (2^5/5^2)^{10} = (32/25)^{10} > 1$ $\therefore (2^{10})^5$ is the largest.

29 D Arithmetic Progression: $(1, 1+d, 1+2d)$ Geometric Progression $(1, r, r^2)$
$1 + 2d = 4 \Rightarrow d = 3/2$; $r^2 = 4$ and $r > 0 \Rightarrow r = 2$. Compute second terms and add.

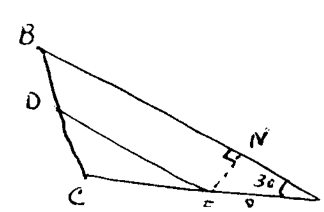
30 B The angle between a tangent and a secant is one-half the difference of the intercepted arcs. $m(\widehat{QT}) = 84°$, $m(\widehat{RT}) = 180°$

31 C For x close to but less than 1 the value of x^{100} is close to zero $\therefore$
$\therefore y^{100} \approx 1 \Rightarrow y$ is very close to 1. The naked eye could not discern the difference between this figure and the square centered at the origin



32 B $1 \le n \le 1000 \Leftrightarrow 1 \le n^2 \le 1{,}000{,}000$ $\therefore$ 1000 perfect squares
$1 \le n \le 100 \Leftrightarrow 1 \le n^3 \le 1{,}000{,}000$ $\therefore$ 100 perfect cubes
which of those counted are both? $1 \le n \le 10 \Leftrightarrow 1 \le n^6 \le 1000000$ thus 10 have been counted twice and thus are 1090 distinct perfect squares and cubes.

33 B $DE = \frac{1}{2}(BA) = 10$. Construct $\overline{EN} \perp \overline{AB}$
$EN = \frac{1}{2} \times$ (hypotenuse of $\triangle ANE$) $= 4$
Apply the formula for the area of a trapezoid.



34 ΔAOD is a 30°-60°-90° triangle with hypotenuse r.
$\therefore OD = r/2$, $AD = \sqrt{3}\,r/2$ and the area of ΔAOB is $\sqrt{3}\,r^2/4$. Multiply this area by three.

l ways.)

-half as long.

27 repeats combinations

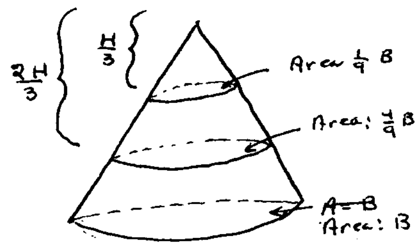
kind, 4 choices for the second, ...

35 E

36 C ~~In order the discriminant $9-4K \geq 0$ for real roots.~~
The discriminant must be non-negative for real roots. $\therefore$
$9-4K \geq 0 \Rightarrow K \leq 2\frac{1}{4}$ or $K \in \{0,1,2\}$. $\therefore$ probability of this event is 3/10.

37. B p pieces at $10/p$ carats each. Cost: $500(p-1)^2$ Selling Price: $1000\left(\frac{10}{p}+1\right)p$
Attached → ~~Profit $F = 1000(10/p+1)p - 500(p-1)^2 = 1000(10+p) - 500(p-1)^2$~~
~~$F = -500p^2 + 2000p + 9500$.~~ Determine the vertex of the associated parabola or use derivatives:
~~$F' = -1000p + 2000$, $F' = 0$ for $p = 2$ (a maximum by F'')~~

38. E Subtract the volume of the top two cones.
$$(1/3)\left(\frac{4}{9}B\right)\frac{2H}{3} - \frac{1}{3}\left(B/9\right)\left(H/3\right) = \frac{BH}{81}(8-1) = \frac{7BH}{81}$$



39 D Expand: $D = y(y-x^2) - x(x-xy) + y(x^2 y^2)$
~~$= y^2 - x^2y - x^2 +$~~
which simplifies and factors as $(y-x)(y+x)(1-y)$

40 D ~~By~~ By DeMoivre's Theorem: $(1+i)^n + (1-i)^n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) +$
$(\sqrt{2})^n\left(\cos\frac{-n\pi}{4} + i\sin\frac{-n\pi}{4}\right)$ ~~which~~ Since Cos and Sin functions are even and odd respectively this simplifies to $2^{n/2}\left(2\cos\frac{n\pi}{4}\right)$
or $2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$