18th Annual Mathematics Prize Competition

Part I Answers

No. Ans. Comments. (Many problems can be solved in several ways.)
1. C
   It would take 1 man 36 hours and two men 18 hours to do

2. B
   1+26, 2+25, ..., 13+14. Beyond one-half of 27 repeats contains

3. A
   5! = 5 * 4 * 3 * 2 * 1 since 5 choices for the first, 4 choices for the second,

4. E
   \( n^4 - n^2 + 3 = 0 \)
   \( (n^2 - 1)(n^2 - 3) = (n+1)(n-1)(n+\sqrt{3})(n-\sqrt{3}) = 0 \)
   Factor.

5. C
   The first sentence implies that the value is dependent only upon the
   weight. Thus, the value of a ton is twice the value of half a ton.

6. A
   \( \log_{10} 3 + \log_{10} 4 + \log_{10} 5 = \log_{10} 60 \)
   \( 3 \times 4 \times 5 = 60 \)

7. C
   1st hour, 2 cells; 2nd hour, 4 cells; ... Nth hour, 2^N cells. Solve: 2^N > 1000
   and choose the least value of N.

8. B
   Each of the 2n houses is connected to the 2n-1 other houses yielding
   2n(2n-1) wires. But this counts each wire twice thus divide by 2. 1/N

9. D
   The number of permutations of the 4 remaining letters is

10. B
   \( \frac{1}{x + y} = \frac{1}{2} + \frac{1}{y} \)
   Apply Quadratic Formula:

11. C
   \( d_1 = 45t \)
   \( d_2 = 105t \)
   \( d_1 + d_2 = 50 \)
   \( 45t + 105t = 50 \)
   \( t = \frac{1}{3} \) hr.

12. A
   Let h = no. of white and (100-h) the number of black: 25h-10(100-h) = -90
   h = 26 and 100-h = 74.

13. A

14. D

15. A
   \( \Delta ABC \sim \Delta DFE \)
   \( \frac{r/p} = \frac{c}{x} \)
   \( \Delta AYX \sim \Delta ADE \)
   \( \frac{r/q} = \frac{AX}{CA} \)
   Add equations and simplify.

16. E
   More than one answer is possible. Solution
   Insufficient information for a unique solution

17. A
   Grouping consecutive pairs of terms, each having value 1, yields
   \( 50(1) - 101 = -51 \)

18. C
   \( 3^{x-1} - 3^{x-1} = 24 \)
   \( \frac{3^x(3-3^1)}{24} \)
   \( x = 2, y = 26 \)

19. E
   Note to satisfy being the reflexive, symmetric, and transitive properties.

20. C
   \( y = x^2 \)
   \( x^2 + (y-1)^2 = 1 \)
21. E

4x^2 - 12x > 0 \iff (x - 3)x > 0 which is satisfied by all values for which both factors are positive or both negative, i.e., x < 0 or x > 3

22. E

|x| > \frac{1}{y} \iff [(x > 0 \text{ and } x > \frac{1}{y}) \text{ or } (x < 0 \text{ and } -x > \frac{1}{y})] \iff (x > 1 \text{ or } x < 0)

23. B

10^{30} = (10^3)^{10} = (998 + 2)^{10} which has the same remainder as 2^{10} since each term of the expansion contains a factor of 988. 2^{10} = 1024, r = 26

24. A

Let n_i represent the number of times the number of the i^th contest won, and (n_1, n_2, n_3) an outcome of the contest. The number of different outcomes of the contest is the number of different permutations for each of (0, 0, 4), (0, 3, 0), (0, 2, 2) and (1, 1, 2) which is 3, 6, 3, and 3 respectively. The sum is 15.

25. B

Solve the three equations simultaneously yielding x = \frac{1}{12}, y = \frac{1}{12}

28. A

\theta_1 is perpendicular to the half plane containing A and thus is perpendicular to every line intersecting plane containing point P, which contains point P and lies in that plane.

29. D

An arithmetic progression: (1, 1 + d, 1 + 2d) Geometric progression (1, r, r^2)

1 + 2d = 4 \iff d = \frac{3}{2}, r^2 = 4 and r > 0 \iff r = 2. Compute second term and add.

30. B

The angle between a tangent and a secant is one-half the difference of the intercepted arcs. m(\overarc{AB}) = 84°, m(\overarc{AB}) = 180°.

31. C

For x close to but less than 1, the value of x^{100} is close to zero. i.e., x^{100} \approx 1 \iff y is very close to 1. The radius y cannot decrease the difference between the figure and the square centered at the origin.

32. B

1 \leq n \leq 1000 \iff 1 \leq n^2 \leq 1000 \iff 100 \text{ perfect squares}

1 \leq n \leq 100 \iff 1 \leq n^2 \leq 10000 \iff 100 \text{ perfect squares}

Which of these centered and both? 1 \leq n \leq 10 \iff 1 \leq n^2 \leq 100 \text{ than 10 have been centered twice and thus an 10 \text{ perfect } squares and above.}

33. B

DE = \frac{1}{2} \overline{AB} = 10. \overline{EN} \perpendicular \overline{AB}

EN = \frac{1}{2} \times \text{ (hypotenuse of } \triangle ANE) = 4

Apply the formula for the area of a triangle.
35 E
36 C
37 B
38 E
39 D
40 D

\[ \triangle ABD \text{ is a } 90^\circ-60^\circ-30^\circ \text{ triangle, with hypotenuse } y. \]
\[ OD = \frac{y}{2}, \quad AD = \frac{\sqrt{3}}{2} y, \quad \text{and the area } P \]
\[ \triangle AOB \text{ is } \frac{\sqrt{3}}{4}. \text{ Multiplying the area by three.} \]

\[ \text{Equation: } \sqrt{y-4x} = 2. \text{ Solve for } y. \]

The discriminant must be non-negative for real roots. \[ 9 - 4k > 0 \implies k < 2 \text{ or } k \in \left[0, \frac{9}{2} \right], \text{ probability of the event is } \frac{3}{10}. \]

37 B
\[ f(x) = \text{ positive at the critical point. } \]
\[ \text{Example: } 500(\frac{1}{2} + 1) \text{ or } 500(\frac{1}{2} - 1) \text{ or } 500(\frac{1}{2} + 1) \text{ or } 500(\frac{1}{2} - 1) \]
\[ f' = 1000(\frac{1}{2} - 1) = 0 \text{ for } x = 2 \text{ (a maximum) or } f'' \]

38 E
\[ \text{Subtract the volume of the top two cones.} \]
\[ (\frac{1}{3})(\frac{7}{2})(\frac{1}{2}) \left[ \frac{1}{3} \left( \frac{7}{2} \right) \right] = \frac{7}{8} \]

39 D
\[ \text{Expand: } D = \frac{7}{8} \left( y - x^2 \right) - \frac{1}{3} \left( \frac{7}{2} \right) \left( \frac{7}{2} - 1 \right) = \frac{7}{8} \]

40 D
\[ \text{By DeMoivre's Theorem: } (1 + i)^n = \left( \sqrt{2} \cos \frac{n\pi}{4} \cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right) \]
\[ \text{or } \sqrt{2} \cos \frac{n\pi}{4} \cos \frac{n\pi}{4} \]

\[ \text{or } \frac{3}{4} \cos \frac{n\pi}{4} \cos \frac{n\pi}{4} \]