EIGHTEENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America with the assistance of Michigan Colleges and Universities, Professional Organizations, and Industries.

PART 1

October 16, 1974

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Check to insure that your six-digit student number has been recorded correctly. Do not make calculations on the answer sheet.

2. Do as many problems as you can in the 100 minutes allowed. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. The first 20 problems of this examination are intended to sample many of the topics in the secondary mathematics curriculum. You may be unfamiliar with some of these topics and quite possibly will find a number of problems which are easier for you distributed throughout the last twenty items. Usually a score of about 20 or more will allow you to become a finalist and write the second exam.

5. In each of the questions five different possible responses are proposed. In some cases the fifth alternative is listed "E" none of these. In such cases if you believe none of the first four alternatives to be correct, mark E.

6. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer or are able to eliminate some of the alternatives as impossible.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. If you have questions concerning the instructions ask them now.

18th ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

1) If three men working at the same rate can paint a barn in 12 hours, then at that rate the number of hours needed for two men to do the job is

A. 8 B. 15 C. 18 D. 24 E. None of these

2) In how many ways, disregarding order, can 27 be written as a sum of two positive integers?

A. 26 B. 13 C. 7 D. 1 E. None of these

3) Five books with differently colored covers are placed upright (left to right) on a shelf of a bookcase. The number of different arrangements possible is

A. 120 B. 60 C. 24 D. 15 E. None of these

4) A class of 20 algebra students meets in a room having 30 seats. Let P be the set of students in the algebra class and Q the set of seats, so $N(P) = 20$ and $N(Q) = 30$. Then

A. $P \subset Q$ B $Q \subset P$ C. $n(P \cup Q) = 30$ D. $P \cap Q = 20$
E. None of these

5) How many roots of $x^4 - 4x^2 + 3 = 0$ are irrational?

A. 0 B. 1 C. 2 D. 3 E. 4

6) Assume that a \$10 gold piece weighs twice as much and is worth twice as much as a \$5 gold piece. If F is the value of a ton of \$5 gold pieces, and T the value of half a ton of \$10 gold pieces, then

A. $F = 2T$ B. $T = 2F$ C. $T = F$ D. $F = 4T$ E. $T = 4F$

7) The value of $10^{\log 3 + \log 4 + \log 5}$, where logarithms are to base ten, is

A. 12 B. 40 C. 60 D. 120 E. None of these

8) A certain type of cell divides into two cells every hour. What is the least whole number of hours it will take to obtain 1000 cells, if one begins with one cell?

A. 5 B. 10 C. 50 D. 100 E. 500

9) An island has $2n$ houses, where n is a positive integer. A primitive phone system consists of separate wires directly connecting each pair of houses. How many such wires are needed?

A. $2n^2 - n$ B. $4n - 3$ C. $2n - 1$ D. n E. $4n^2 - 2n$

10) Among all the distinct arrangements of the five letters S,P,E,L,L, how many begin with the letter L?

A. 120 B. 84 C. 36 D. 24 E. None of these

11) If $(2+x)^{-1} = 2^{-1} + x^{-1}$, then

A. x is any real number B. $x = -1 \pm i\sqrt{3}$ C. $x = -1 \pm i$

D. $x = 1$ E. $x = -2$

12) Two trains travel toward each other at speeds of 45 mph and 105 mph respectively. At 5:00 p.m. they are 50 miles apart. At what time do they meet?

A. 5:10 B. 5:15 C. 5:20 D. 5:30 E. None of these

13) In a shooting contest a man receives a quarter each time he hits the target, and pays a dime each time he misses. If he has lost 90 cents after taking 100 shots, how many times has he missed the target?

A. 74 B. 60 C. 42 D. 9 E. None of these

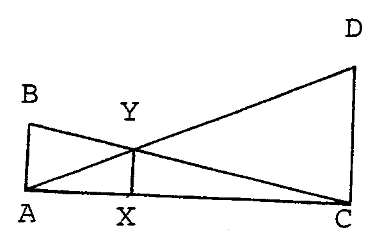
14) In base eight notation, the integer which is two greater than the base eight number 766 is written as

A. 1376 B. 768 C. 777 D. 770 E. None of these

15) Line segments $\overline{AB}$, $\overline{XY}$, and $\overline{CD}$ are each perpendicular to $\overline{AC}$, and Y is the intersection of $\overline{AD}$ and $\overline{CB}$. If $AB = p$, $CD = q$, and $XY = r$, then $r/p + r/q$ equals

A. 1 B. $\frac{1}{2}$ C. 2 D. $\frac{3}{2}$

E. None of these



16) Sets P,Q and R each contain 5 elements. If $P \cap Q$, $Q \cap R$ and $R \cap P$ each contain 3 elements, how many elements has $P \cap Q \cap R$?

A. 0 B. 1 C. 2 D. 3 E. Not enough information

17) The value of $\sum_{n=1}^{101} (-1)^n n = -1 + 2 - 3 + \cdots + 100 - 101$ is

A. -51 B. -49 C. -1 D. 0 E. 50

18) Solve $3^{x+1} - 3^{x-1} = 24$ for the value of x. What is the corresponding value of y if $y = (x+1)^3 - (x-1)^3$?

A. $y = 8$ B. $y = 36$ C. $y = 26$ D. $y = 1/24$
E. $y = 18$

19) Which of the following binary relations defined on the set of all people is an equivalence relation?

A. "lives within 10 miles of" B. "was born within a month of"
C. "is married to" D. "has met" E. None of these

20) The curves $y = x^2$ and $y^2 + (x-1)^2 = 1$ intersect at how many points?

A. 0 B. 1 C. 2 D. 3 E. Infinitely many

21) If $4x - 3x^2 < x^2 - 8x$, it must be true that

A. $x < 0$ B. $0 < x < 3$ C. $x > 3$ D. $x < 3$
E. None of these

22) The inequality $|x| > \frac{1}{x}$ has the solution

A. $|x| < 1$ B. $x^2 < 1$ C. $0 < x < 1$ D. $x < 1$

E. $x < 0$ or $x > 1$

23) The contrapositive of the statement "If Socrates is mortal, then he is a man.", is

A. If Socrates is a man, then he is not mortal

B. If Socrates is not a man, then he is not mortal

C. If Socrates is not mortal, then he is not a man

D. If Socrates is a man, then he is mortal

E. None of these

24) What is the remainder when 10^{30} is divided by 998?

A. 2 B. 26 C. 498 D. 997 E. None of these

25) Four cash prizes of equal value are divided among three different contestants on a quiz program, where each contestant does not necessarily win a prize. If each prize is won intact by one of the contestants, how many different outcomes of this contest are possible?

A. 15 B. 24 C. 18 D. 3 E. 4

26) If b is properly chosen, the lines $y = x$, $y = 2x + b$ and $y = bx - 1/4$ intersect at one point. This point is

A. (0,0) B. $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ C. (1,-1) D. (4,4) E. (1,1)

27) Which of the following represents the largest number?

A. $10!$ B. 10^{10} C. 9876543210 D. $(2^{10})^5$

E. $(5^{10})^2$ (Note: $n! = 1 \cdot 2 \cdot 3 \cdots n$)

28) Two half planes with the common edge $\overleftrightarrow{PE}$ are perpendicular. The point L is in one half plane with $\overrightarrow{PL}$ perpendicular to $\overleftrightarrow{PE}$. The point Q is in the other half plane and angle QPE measures 30°. The measure of angle QPL is

A. 90° B. 120° C. Between 90° and 120° D. 150°

E. None of these

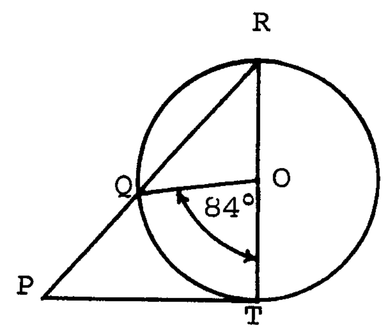
29) An arithmetic progression has three terms and a geometric progression of positive numbers also has three terms. If both start with 1 and end with 4, the sum of the second terms is

A. 3 B. $3\frac{1}{2}$ C. 4 D. $4\frac{1}{2}$ E. None of these

30) Line PT is tangent to the circle with center at O at point T. Line RT is a diameter and PR cuts the circle at Q. If the measure of angle TOQ = 84°, then the measure of angle TPQ is

A. 60° B. 48° C. 45°

D. 42° E. None of these



31) Let S be the set of ordered pairs of reals defined by $S = \{(x,y) \mid x^{100} + y^{100} = 1\}$. Which shape most accurately describes the cartesian graph of S?

A. 4 points B. a triangle C. a square D. a circle

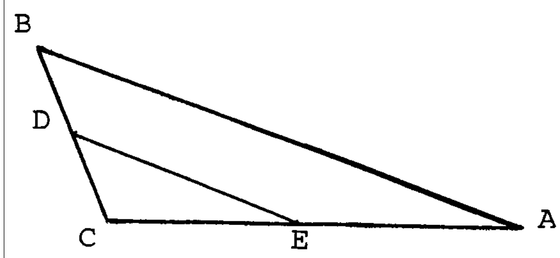
E. an ellipse

32) The number of integers from 1 to 1 million which are not perfect squares or perfect cubes is

A. 999,000 B. 998,910 C. 998,900 D. 998,890

E. None of these

33) In triangle ABC, points D and E are midpoints of sides $\overline{CB}$ and $\overline{CA}$, respectively. Also the measure of angle BAC = 30°, AB = 20, and AC = 16. The area of quadrilateral ABDE is



A. 40 B. 60 C. 72 D. 80 E. None of these

34) An equilateral triangle is inscribed in a circle of radius R. The triangle has area

A. $3R^2\sqrt{3}/4$ B. $R^2/3$ C. $3R^2$ D. $3R^2/2$

E. None of these

35) The intersection of a tetrahedron and a plane can

A. Never be a point

B. Never be a line segment

C. Never be a triangle

D. Never be a quadrilateral

E. None of the above is correct

36) The integer k is selected at random from the set of ten integers {0,1,2,...,9}. What is the probability that the equation $x^2 + 3x + k = 0$ has real roots?

A. 0 B. 1/10 C. 3/10 D. 1/3 E. 1

37) It costs $\not{\$}500(p-1)^2$ to cut a diamond into p pieces. If a k-carat diamond (by weight) sells for $\not{\$}1000(k+1)$, where $k \geq 1$, in how many pieces should one cut a 10 carat diamond to obtain the maximum profit?

A. 1 B. 2 C. 3 D. 4 E. None of these

38) A cone of height H and base area B is cut into three pieces by planes parallel to the base. If the planes are at heights $H/3$ and $2H/3$ above the base, the volume of the piece of the cone between the two planes is

A. $BH/9$ B. $BH/27$ C. $BH/81$ D. $4BH/27$ E. $7BH/81$

39) If the determinant $\begin{vmatrix} y & x & y \\ x & y & x \\ y & x & 1 \end{vmatrix}$ is zero, it must be true that

A. $y = 1$ B. $x = y$ C. $x = -y$ D. $y = 1$ or $x = y$ or $x = -y$

E. None of these

40) If $i^2 = -1$ and n is a positive integer then $(1+i)^n + (1-i)^n$ equals

A. 2^n B. $2^{(n/2)+1}$ C. $\cos \frac{n\pi}{4}$ D. $2^{\frac{n+2}{2}} \cos \frac{n\pi}{4}$

E. None of these

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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