17th Annual Michigan Mathematics Prize Competition
Sample Solutions for Part II

The sketch of one solution for each of the Part II problems is given below. Solutions do not include possible generalizations.

1. Solve the system of equations

$xy = 2x + 3y$

$yz = 2y + 3z$

$zx = 2z + 3x$

Solution: $x = 0 \Rightarrow y = 0$ and $z = 0$ and thus $x = y = z = 0$ is a solution to the system.

$x \neq 0 \Rightarrow y \neq 0$ and $z \neq 0$ so that we may divide through each equation by the nonzero quantity on its left side to yield:

$$1 = \frac{2}{y} + \frac{3}{x}$$

$$1 = \frac{2}{z} + \frac{3}{y}$$

$$1 = \frac{2}{x} + \frac{3}{z}$$

Solving this system simultaneously for x, y, and z we obtain: $x = y = z = 5$.

Thus the solution set of the original system is given by: $\{(0,0,0),(5,5,5)\}$

2. For any integer k greater than 1 and any positive integer n, prove that n^k is the sum of n consecutive odd integers.

Solution: Let k be any integer greater than 1 and $n \geq 2$. (result obvious for $n = 1$)
Then $k - 1 \geq 1$ and:

$$n^k = n \cdot n^{k-1} = n^{k-1} + n^{k-1} + \cdots + n^{k-1} \text{ (n terms)}$$

Case 1: Let $n = 2s + 1$ (odd) $\Rightarrow n^{k-1}$ is odd. The sum of the sequence:

$2s,\ 2s - 2,\ 2s - 4,\ \ldots,\ 2s - (n-1)2$ is zero since

$$\sum_{m=0}^{n-1}(2s - 2m) = \sum_{m=0}^{n-1} 2s - 2\sum_{m=0}^{n-1} m = n(2s) - 2\left[\frac{n(n-1)}{2}\right] = n(2s) - (n-1)n$$

$= n(n-1) - (n-1)n = 0$ since $2s = n-1$.

Using this n termed sequence of consecutive even integers with the expansion of n^k above yields:

$$n^k = \left[n^{k-1} - 2s\right] + \left[n^{k-1} - (2s - 2)\right] + \cdots + \left[n^{k-1} - (2s - (n-1)2)\right]$$

as desired.

Case 2: Let $n = 2s \Rightarrow n^{k-1}$ is even. We now use the following sequence of consecutive odd integers whose sum is also zero:

$2s - 1,\ 2s - 3,\ \ldots,\ 2s - \left[(n-1)2 + 1\right]$ together with the above expansion for n^k to provide the desired representation:

$$n^k = \left[n^{k-1} - (2s - 1)\right] + \left[n^{k-1} - (2s - 3)\right] + \cdots$$

to the system.

$x \neq 0 \Rightarrow y \neq 0$ and $z \neq 0$ so that we may divide through each equation by the nonzero quantity on its left side to yield:

$$1 = \frac{2}{y} + \frac{3}{x}$$

$$1 = \frac{2}{z} + \frac{3}{y}$$

$$1 = \frac{2}{x} + \frac{3}{z}$$

Solving this system simultaneously for x, y, and z we obtain: $x = y = z = 5$.

Thus the solution set of the original system is given by: $\{(0,0,0),(5,5,5)\}$

2. For any integer k greater than 1 and any positive integer n, prove that n^k is the sum of n consecutive odd integers.

Solution: Let k be any integer greater than 1 and $n \geq 2$. (result obvious for $n = 1$)
Then $k - 1 \geq 1$ and:

$$n^k = n \cdot n^{k-1} = n^{k-1} + n^{k-1} + \cdots + n^{k-1} \text{ (n terms)}$$

Case 1: Let $n = 2s + 1$ (odd) $\Rightarrow n^{k-1}$ is odd. The sum of the sequence: $2s, 2s - 2, 2s - 4, \ldots, 2s - (n-1)2$ is zero since

$$\sum_{m=0}^{n-1} (2s - 2m) = \sum_{m=0}^{n-1} 2s - 2\sum_{m=0}^{n-1} m = n(2s) - 2\left[\frac{n(n-1)}{2}\right] = n(2s) - (n-1)n$$

$= n(n-1) - (n-1)n = 0$ since $2s = n-1$.

Using this n termed sequence of consecutive even integers with the expansion of n^k above yields:

$$n^k = \left[n^{k-1} - 2s\right] + \left[n^{k-1} - (2s - 2)\right] + \cdots + \left[n^{k-1} - (2s - (n-1)2)\right]$$

as desired.

Case 2: Let $n = 2s \Rightarrow n^{k-1}$ is even. We now use the following sequence of consecutive odd integers whose sum is also zero:

$2s - 1, 2s - 3, \ldots, 2s - \left[(n-1)2 + 1\right]$ together with the above expansion for n^k to provide the desired representation:

$$n^k = \left[n^{k-1} - (2s - 1)\right] + \left[n^{k-1} - (2s - 3)\right] + \cdots + \left[n^{k-1} - (2s - ((n-1)2 + 1))\right]$$

3. Determine all pairs of real numbers, x_1, x_2 with $|x_1| \leq 1$ and $|x_2| \leq 1$ which satisfy the inequality:
$|x^2 - 1| \leq |x-x_1||x-x_2|$ for all x such that $|x| \geq 1$.

Solution: The restrictions on x and the x_i permit the deletion of the absolute value signs as follows:

$$|x| \geq 1 \Leftrightarrow x^2 \geq 1 \Leftrightarrow x^2 - 1 \geq 0 \Rightarrow |x^2 - 1| = x^2 - 1$$

Separate consideration of the cases $x \leq -1$ and $x \geq 1$ will further show that $|x - x_1||x - x_2| = (x - x_1)(x - x_2)$.

Therefore $|x^2-1| \le |x-x_1||x-x_2| \Leftrightarrow x^2-1 \le (x-x_1)(x-x_2) \Leftrightarrow$ $(x_1 + x_2)x \le 1 + x_1x_2$. Since this inequality on the x_i is to be true for all $|x| \ge 1$ we have: $x_1 + x_2 = 0$. The desired set of pairs is $\{(x,-x) \mid -1 \le x \le 1\}$

4. Find the smallest positive integer having exactly 100 different positive divisors. (The number 1 counts as a divisor).

Solution: The desired number can be written as: $2^\alpha \times 3^\beta \times 5^\gamma \times 7^\delta \times \ldots$ Since for each prime p, p^n provides n+1 different sets of factors corresponding to those containing 0,1,2,...,n factors of p and the total number of factors is to be 100, we have:

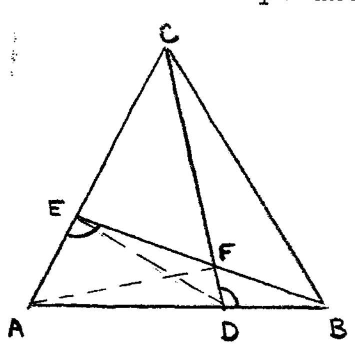
$$(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)\cdots = 100$$

Factorizations of 100	Corresponding Numbers
100 x 1	2^{99}
50 x 2	$2^{49} \times 3$
20 x 5	$2^{19} \times 3^4$
25 x 4	$2^{24} \times 3^3$
10 x 10	$2^9 \times 3^9$
25 x 2 x 2	$2^{24} \times 3 \times 5$
10 x 5 x 2	$2^9 \times 3^4 \times 5$
5 x 5 x 4	$2^4 \times 3^4 \times 5^3$
5 x 5 x 2 x 2	$2^4 \times 3^4 \times 5 \times 7$

The smallest of these is $2^4 \times 3^4 \times 5 \times 7 = 45360$

5. ABC is an equilateral triangle of side 3 inches. DB = AE = 1 in. and F is the point of intersection of segments $\overline{CD}$ and $\overline{BE}$. Prove that $\overline{AF} \perp \overline{CD}$.

Solution: We will show that quadrilateral ADFE is an inscribed quadrilateral with $\overline{AD}$ the diameter of the circumcircle.



Thus $\angle$ AFD is inscribed in a semicircle.

$\Delta ABE \cong \Delta BCD$ by S.A.S. Theorem
$\therefore \angle AEB \cong \angle CDB \therefore$ The $\angle D$ and $\angle E$ of quadrilateral ADFE are supplementary so that quadrilateral ADFE is an inscribed quadrilateral. Applying the law of cosines to ΔADE we have:

$$DE^2 = 1^2 + 2^2 - 4\cos 60^\circ = 5 - 2 = 3$$

$\therefore DE = \sqrt{3}$ and the Pythagorean relation justifies that the inscribed $\Delta\overline{ADE}$ is a right triangle with right angle at E so that $\overline{AD}$ is a diameter of the circumcircle. Thus, $\angle$ AFD is a right angle and AF $\perp$ CD as desired.