17th ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Part I Answers

No.	Ans.	Comments. (Many problems can be solved in several ways.)
1	C	Use proportions from the similar triangles.
2	E	Apply the Pythagorean Theorem twice, first to find the diagonal of the base and then the diagonal of the cube.
3	C	The number of all subsets of $\{1,2,4,5\}$ is $2^4 = 16$. Adding 3 to each we get 16.
4	C	Cut and count.
5	C	Simplify inside brackets first using $x^{-1} = \frac{1}{x}$.
6	B	$\sqrt{3-x} = x\sqrt{3-x} \Longleftrightarrow \sqrt{3-x}\,(1-x) = 0 \therefore x=1$ or $x=3$.
7	A	$x^2 + Ax + 1 = (x - r_1)(x - r_2) \Rightarrow r_1 + r_2 = -A$ and $r_1 r_2 = 1$ $\lvert r_1\rvert + \lvert r_2\rvert \geq \lvert r_1 + r_2\rvert = A > 2$ and $\lvert r_1\rvert\lvert r_2\rvert = 1 \therefore$ exactly one root has absolute value less than 1.
8	D	$4s = 1$, $2\pi r = 1 \therefore s = \frac{1}{4}$ and $r = \frac{1}{2\pi}$. Substitute into $s^2 + \pi r^2$.
9	C	Shaded area $= 64 - 4 \cdot \frac{\pi \cdot 4^2}{4}$.
10	C	$30^2 = 60(60 - 2r)$
11	C	Use Law of Cosines; $a^2 = b^2 + c^2 - 2bc \operatorname{Cos} \alpha$, and $\operatorname{Cos} 60^\circ = \frac{1}{2}$.
12	D	Reciprocal of abscissas of intersection of the unit circle, $x^2 + y^2 = 1$ and $y = -\frac{1}{2}$.
13	B	i), iii), iv) are correct
14	D	Solve over two subdomains of x. CaseI: $x < 2 \therefore x - 2 < 0$ and $\lvert x - 2\rvert = 2 - x$. CaseII: $x \geq 2 \therefore x - 2 \geq 0$ and $\lvert x - 2\rvert = x - 2$.
15	A	Let $a + bi = \sqrt{-7 - 24i}$. Square both sides, equate real and imaginary parts and solve for a and b.
16	E	$\frac{4}{3}\pi r^3 = \pi r^2 h \therefore h = \frac{4r}{3}$
17	A	Number of ways of drawing 2 white out of 6 is $\binom{6}{2} = 15$ Number of ways of drawing 2 black out of 4 is $\binom{4}{2} = 6$ Number of ways of drawing 4 balls out of 10 is $\binom{10}{4} = 210$ $\therefore$ probability of drawing any one group of 4 is 1/210. There are 15 x 6 desirable such combinations $\therefore p = 15 \times 6 \times (1/210) = \frac{3}{7}$.
18	E	Solve $\pi(r + 1)^2 = 2(\pi r^2)$ for r.

5	C	Simplify inside brackets first using $x^{-1} = \frac{1}{x}$.
6	B	$\sqrt{3-x} = x\sqrt{3-x} \Longleftrightarrow \sqrt{3-x}\,(1-x) = 0 \therefore x=1$ or $x=3$.
7	A	$x^2 + Ax + 1 = (x - r_1)(x - r_2) \longrightarrow r_1 + r_2 = -A$ and $r_1 r_2 = 1$ $\lvert r_1\rvert + \lvert r_2\rvert \geq \lvert r_1 + r_2\rvert = A > 2$ and $\lvert r_1\rvert\lvert r_2\rvert = 1 \therefore$ exactly one root has absolute value less than 1.
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12	D	Reciprocal of abscissas of intersection of the unit circle, $x^2 + y^2 = 1$ and $y = -\frac{1}{2}$.
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18	E	Solve $\pi(r + 1)^2 = 2(\pi r^2)$ for r.
19	A	The discriminant must be zero $\therefore 64k^2 - 4 \cdot 8(3k + 2) = 0$. Solve for k.
20	B	$f(x) = x + \frac{1}{x} = (\sqrt{x} - 1/\sqrt{x})^2 + 2 \geq 2 \therefore$ no number less than 2.
21	A	Determinant is $x^2 - 2x + 1 \therefore$ Solve $x^2 - 2x \leq 0$ for x.* (See Footnote)
22	C	Solve $10 = \frac{r}{1-r}$ where r is the first term.

No.	Ans.	Comments (Many problems can be solved in several ways.)
23	A	Without Trig: $m\angle CAD = 30^\circ$. Drop perpendiculars from D to $\overline{AC}$ and $\overline{BC}$ and use 30°-60°-90° triangles. With Trig: Extend $\overline{CD}$ to E on $\overline{AB}$. ΔAEC is right isosceles ∴ $AE = 2\sqrt{2}\,\text{Cos } 15^\circ$ and AC = BC. $\text{Cos } 15^\circ = \sqrt{\frac{1+\text{Cos } 30^\circ}{2}} = \sqrt{\frac{1+(\sqrt{3}/2)}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2} = \frac{\sqrt{\left(\frac{\sqrt{3}}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)^2}}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$ $BC = \sqrt{3}+1$. Note: To complete the square under the radical above set $(a+b)^2 = 2+\sqrt{3}$ ---yields $a^2+2ab+b^2 = 2+\sqrt{3}$ set $a^2+b^2 = 2$, $2ab = \sqrt{3}$ and solve for a and b.
24	B	Expand or $(1+i)^4 = (\sqrt{2}e^{\frac{\pi}{4}i})^4 = 4e^{\pi i} = 4(-1) = -4$.
25	D	Ratio of areas is equal to the ratio of the squares of the radii.
26	D	
27	D	Equivalent to: Distance (x,y) is from (-1,0) added to its distance from (1,0) is 2 units. ∴ $(x,y) \in [-1,1]$ on the x axis since (-1,0) and (1,0) are 2 units apart.
28	B	$10x = 4.\overline{91}$ $1000x = 491.\overline{91}$ ∴ $990x = 487$.
29	B	Extend altitude to circle and use theorem concerning the product of segments of intersecting chords together with measure of radius.
30	D	measures of intercepted arcs must be in the ratio 2:3:4 and $C = 30\pi$ ∴ solve $2x + 3x + 4x = 30\pi$ for x and compute the desired multiples of x.
31	E	$\frac{1}{2}$ is a root of the polynomial ∴ $x - \frac{1}{2}$ is a factor.
32	A	Construct $\overline{AC}$. QP and MN are each $\frac{1}{2}AC$. Repeat with $\overline{BD}$.
33	C	D - distance, D/3 - speed of express train, 2D/9 - speed of slow train, t - time in hrs. after 3PM $\frac{Dt}{3} + \frac{2D}{9}\left(t + \frac{3}{2}\right) = D$.
34	E	Substitute $1 + \log_3 X = \log_3 3 + \log_3 x = \log_3 3x$, take antilogs, solve.
35	E	In one minute the minute hand generates an angle of 6°, in one minute the hour hand generates an angle of $\frac{1}{2}^\circ$, t - time in minutes after 4 PM, ∴ $6t - 120^\circ - \frac{t}{2} = 78^\circ$.
36	B	$\log(7.5^3)^{16} = 16[\log 7 + 3\log 5] = 47.0736$ ∴ 48 digits.
37	D	Integral solutions for $0 \le d \le 50$, $0 \le q \le 20$ in $10d + 25q = 500$ or $q = 20 - (2/5)d$ ∴ $d = 5t$, $q = 20 - 2t$ and $t = 0,1,\ldots,10$.
38	E	Solve in four cases corresponding to restricted domains $x<-1$, $-1\le x<0$, $0\le x<1$, $x\ge 1$, allowing replacement of $\|a\|$ by a if $a\ge 0$ and by $-a$ if $a<0$.
39	B	Regular hexagon has greatest area $A = 6 \cdot 2 \cdot 2\sqrt{3}$.
40	A	$10^2 \equiv 1 \pmod{99}$ ∴ $10^{44} \equiv 1 \pmod{99}$.

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* The answer "B" in problem 21 is also logically correct. Scores of students in contention for attending the Awards Program who chose "B" will be adjusted before award recipients are selected.