SEVENTEENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America, Michigan Colleges and Universities, Professional Organizations, Industries, and Foundations.

PART 1

October 17, 1973

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Check to insure that your six-digit student number has been recorded correctly. Do not make calculations on the answer sheet.

2. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. The first 20 problems of this examination are intended to sample many of the topics in the secondary mathematics curriculum. You may be unfamiliar with some of these topics and quite possibly will find a number of problems which are easier for you distributed throughout the last twenty items. Usually a score of about 20 or more will allow you to become a finalist and write the second exam.

5. In each of the questions five different possible responses are proposed. In some cases the fifth alternative is listed "E" none of these. In such cases if you believe none of the first four alternatives to be correct, mark E.

6. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer or are able to eliminate some of the alternatives as impossible.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. If you have questions concerning the instructions ask them now.

17th ANNUAL MICHIGAN MATHEMATICS

PRIZE COMPETITION

1. A man 6 ft. tall stands 8 ft. from a lamp-post. If his shadow is 16 ft. long, then the height of the lamp is

   A. 12 B. 10 C. 9 D. 14 E. 11

2. A fly is in one corner of a cubical room 10 × 10 × 10. What is its shortest distance to the opposite corner?

   A. $10\sqrt{5}$ B. 30 C. $10 + 2\sqrt{10}$ D. $10\sqrt{10}$ E. $10\sqrt{3}$

3. A set has 5 elements numbered from 1 to 5. How many subsets contain the element numbered 3?

   A. 32 B. 31 C. 16 D. 8 E. None of these

4. The outer surface of a brick 3 inches by 4 inches by 5 inches is painted blue. If it is cut into one inch cubes, how many of the cubes are painted blue on exactly one side?

   A. 50 B. None C. 22 D. 30 E. None of these

5. When simplified, the expression

$$[a^{-1}+b^{-1} - 2(a+b)^{-1}]^{-1}$$

   is equal to

   A. ab B. $-(a + b)$ C. $\frac{(ab)(a+b)}{a^2+b^2}$ D. $\frac{(a^2 + b^2)ab}{a + b}$

   E. None of these

6. The number of roots satisfying the equation $\sqrt{3-x} = x\sqrt{3-x}$ is:

A. 3 B. 2 C. 1 D. 0 E. Unlimited

7. In the quadratic equation $x^2 + Ax + 1 = 0$, A is a complex number of absolute value greater than 2. The number of roots of absolute value less than 1 is

A. Exactly 1 B. Exactly 2 C. 0 or 1

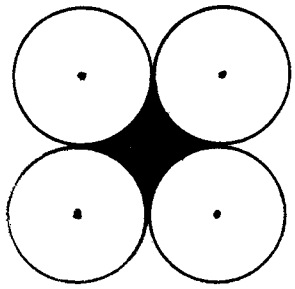
D. 1 or 2 E. None of these

8. A square and a circle each has perimeter 1. The sum of these areas is

A. $\frac{1}{8}$ B. $\frac{1}{2\pi}$ C. $\frac{1}{8} + \frac{1}{2\pi}$ D. $\frac{1}{16} + \frac{1}{4\pi}$

E. None of these

9. In the diagram the circles having radius 4, are tangent as indicated, and their centers are the vertices of a square. What is the area of the shaded part.

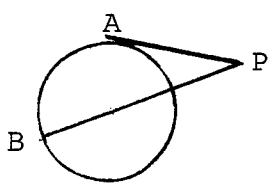


A. 64 B. 16π C. $16(4-\pi)$

D. $64 + 16\pi$ E. None of these

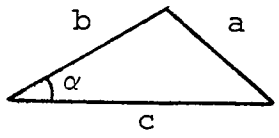
10. The length of a tangent segment from a point P to a circle is 30 in. and that of the segment $\overline{PB}$ through the center is 60 in. The radius of the circle is

A. 25 B. 23 C. $22\frac{1}{2}$ D. $27\frac{1}{2}$ E. 26



11. In a triangle $b = 5$, $c = 8$ and $\alpha = 60°$.

Then a is equal to



A. $20\sqrt{3}$ B. $20\sqrt{2}$ C. 7

D. 6 E. 9

12. If $\sin\theta = -1/2$, then one of the possible values for $\sec\theta$ is

A. $\frac{1}{2}$ B. 2 C. $-\frac{\sqrt{2}}{3}$ D. $\frac{2}{\sqrt{3}}$ E. None of these

13. Consider the following statements where n is a positive integer.

i) n^2 cannot be a prime

ii) $n^2 + 1$ cannot be a prime

iii) $n^2 + 2n$ cannot be a prime

iv) $n^2 + 2n + 1$ cannot be a prime

A. All are correct B. Exactly three are correct

C. Exactly two are correct D. Exactly one is correct

E. None are correct

14. The inequality $|x-2| > x + 1$ is true for all x satisfying

A. $2 < x < 4$ B. $x \geq \frac{1}{2}$ C. $\frac{1}{2} < x < 2$

D. $x < \frac{1}{2}$ E. None of these

15. The square roots of the complex number $-7 - 24i$ are

A. $\pm(3-4i)$ B. $\pm(25+i)$ C. $3 \pm 4i$ D. $\sqrt{7} \pm 2\sqrt{3}i$

E. $4 \pm 3i$

16. A spherical flask of radius r is filled with water. This is then emptied into a right cylindrical flask (with flat bottom) of the same radius. To what height does the water rise?

A. $\frac{4}{3}\pi r$ B. $\frac{1}{3}\pi r$ C. $\frac{1}{3}\pi r^2$ D. $\frac{1}{3}r$ E. $\frac{4}{3}r$

17. If four balls are drawn at one trial from a bag containing six white and four black balls, then the probability that two white and two black balls will be drawn is

A. $\frac{3}{7}$ B. $\frac{1}{2}$ C. $\frac{1}{210}$ D. $\frac{2}{15}$ E. None of these

18. If increasing the radius of a circle by 1 foot doubles its area then the radius is:

A. $r + \sqrt{2}$ B. $\sqrt{2} - 1$ C. $2r$, r any positive number

D. $\sqrt{2}\, r$, E. $1 + \sqrt{2}$

19. For the equation $8x^2 + 8kx + 3k + 2 = 0$ to have equal roots k must be:

A. $2, -\frac{1}{2}$ B. 0 C. 18, - 1 D. 1.

E. No such value of k

20. Define the function $f(x) = x + \frac{1}{x}$ where the domain is the set of all positive real numbers.

A. The range contains no integers

B. The range contains no number less than 2

C. The range contains no number greater than 100

D. Both A) and B) are true

E. All of the above are true

21. For what values of x is it true that

$$\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & x \end{vmatrix} \leq 1$$

A. $0 \leq x \leq 2$ B. $0 \leq x \leq 1$ C. $x \leq 0$ D. $-1 \leq x \leq 1$

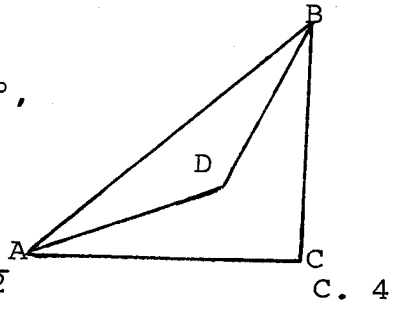
E. None of these

22. The sum of an infinite geometric progression is 10. The first term and ratio are equal. The first term is

A. $\frac{1}{10}$ B. $\frac{1}{11}$ C. $\frac{10}{11}$ D. $\frac{11}{10}$ E. None of these

23. In the figure, $m\angle C = 90°$, $m\angle CAD = 30°$, $m\angle CAB = 45°$, $m\angle ADB = 150°$ $AD = 2$.

What is BC?



A. $1 + \sqrt{3}$ B. $\sqrt{2}$ C. 4

D. Insufficient information to determine E. None of these

24. The complex number $(1+i)^4$ equals

A. $-6 + 0i$ B. $-4 + 0i$ C. $-4 + 8i$ D. $4 + 0i$

E. $0 + 0i$

25. Regular pentagons are inscribed in two circles of radii 2 in. and 5 in. respectively. What is the ratio of their areas?

A. $\frac{4}{25} \cdot \sin 72°$ B. $\frac{2}{5}$ C. $\frac{2}{5} \cdot \sin 72°$ D. $\frac{4}{25}$

E. None of these

26. The region in the coordinate plane defined by the following conditions $x > 0$, and $y < 0$, and $x + y < 0$ is the interior of

A. A half plane
B. An obtuse angle
C. A right angle
D. An acute angle
E. None of these

27. The points in a coordinate plane which satisfy $\sqrt{(x-1)^2+y^2} + \sqrt{(x+1)^2+y^2} = 2$ form

A. A non-degenerate ellipse
B. A circle
C. A hyperbola
D. A line segment
E. None of these

28. If $0.4919191\cdots$ is expressed as the quotient of two integers it would equal:

A. $\frac{491}{1000}$ B. $\frac{487}{990}$ C. $\frac{397}{990}$ D. $\frac{45}{330}$ E. None of these

29. In an isosceles triangle the sum of its base and its height is 10 in. and the radius of the circumscribed circle is $\frac{25}{8}$ in. The sides of the triangle are

A. $5\sqrt{2}$, $5\sqrt{2}$, 5
B. $\frac{5\sqrt{5}}{2}$, $\frac{5\sqrt{5}}{2}$, 5 or 5,5,6
C. 4, 4, 5 or $5\sqrt{3}$, $5\sqrt{3}$, 4
D. 5,5,5 or 6,6,6
E. 8,8,6 or 4,4,5

30. A triangle is inscribed in a circle of radius 15 in. If the measures of the angles of the triangle are in the ratio 2: 3: 4, then the lengths of the corresponding arcs are

A. 5π, 15π, 10π B. 8π, 10π, 12π C. $\frac{4\pi}{9}$, $\frac{2\pi}{3}$, $\frac{8\pi}{9}$

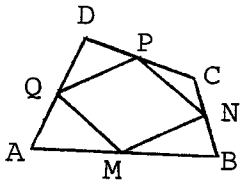
D. $\frac{20\pi}{3}$, 10π, $\frac{40\pi}{3}$ E. None of these

31. Which of the following is a factor of $32x^{10} - 33x^5 + 1$:

A. $x^2 - x - 2$ B. $x + \frac{1}{2}$ C. $x + 1$ D. $x^2 - x - 6$

E. $x - \frac{1}{2}$

32. The quadrilateral ABCD has its sides bisected at M,N,P,Q. If AC = 16, BD = 26, the perimeter of MNPQ is:



A. 42 B. 21 C. 10 D. 20 E. None of these

33. An express train leaves Detroit at 3:00 PM and reaches Niles at 6:00 PM. The slow train leaves Niles at 1:30 PM and arrives at Detroit at 6:00 PM. If both trains travel at constant speeds at what time do they meet?

A. 4:48 PM B. 3:18 PM C. 4:12 PM D. 4:30 PM

E. Insufficient information

34. The solution of $\log_3(1+x) = 1 + \log_3 x$ is:

A. 3 B. All real numbers C. All positive real numbers

D. No solution E. $\frac{1}{2}$

35. At what time between 4 and 5 o'clock will the minute hand of a watch be 13 minutes ahead of the hour hand?

A. 4:30 B. 4:39 C. 4:49 D. 4:38:05 E. 4:36

36. Find the number of digits in 875^{16} given $\log 5 = .6990$ $\log 7 = .8451$

A. 49 B. 48 C. 47 D. 43 E. None of these

37. Using only dimes and quarters, in how many ways can change for a $5 bill be made?

A. 9 B. 20 C. 50 D. 11 E. None of these

38. The solution set of $|x-1| - 2|x| = 3|x+1|$ contains

A. Exactly three elements B. Exactly two elements

C. More than 3 elements D. No elements

E. Exactly one element

39. One has six straws of length four inches each. What is the greatest possible area enclosed inside a planar polygon consisting of these straws?

A. $32 + 8\sqrt{2}$ sq. in. B. $24\sqrt{3}$ sq. in. C. 96 sq. in.
D. 48 sq. in. E. $4\pi^2 - 6\pi/4^2$ sq. in.

40. What is the remainder when 10^{44} is divided by 99?

A. 1 B. 43 C. 98 D. $10^{42} + 1$ E. None of these

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