16th ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Part I Answers

No.	Ans.	Comments. (Many problems can be solved in several ways.)
1	A	$x^2 = 16\pi$, $\therefore 4x/(8\pi) = 2/\sqrt{\pi}$.
2	C	
3	A	$(x + 1)^2 = x^2 + 9$, $\therefore x = 4$.
4	D	
5	D	$2(\pi r^2) = \pi(3/2)^2$, $r = 3\sqrt{2}/4$, $d = 3\sqrt{2}/2$.
6	C	$100 + 2(x + .4x) = 172$, $x = 180/7$.
7	E	Remember that 2 is a prime.
8	E	$7^9 + 4 = (2 + 5)^9 + 4 = 2^9 + 9 \cdot 2^8 \cdot 5 + \cdot \cdot \cdot + 5^9 + 4$ $= 2^9 + 5(*) + 4 = 516 + 5(*) = 1 + 5 \cdot 103 + 5(*)$. $\therefore$ rem $= 1$. Or use congruences mod 5.
9	E	Solve by factoring.
10	A	By division, or by substitution of $x = -1$.
11	D	Remember that 0 is even.
12	D	The 3rd equation = $3\cdot$(first) $- 4\cdot$(second). Or observe determinant = 0, ($\therefore$ C or D), and find one particular solution, such as $(0, 1/5, 1/5)$.
13	A	$\lvert 2x + 1\rvert = \pm(2x + 1)$ depending on $x \geq -1/2$ or $x \leq -1/2$; $\lvert x - 4\rvert = \pm(x - 4)$ depending on $x \geq 4$ or $x \leq 4$. $\therefore$ Solve separately in the three intervals $(x \leq -1/2, -1/2 \leq x \leq 4, x \geq 4)$ and obtain $x = -3, 5$.
14	C	
15	B	$f(x_2) = f(x_1)$ only if $(ax_2 + b)(cx_1 + d) = (ax_1 + b)(cx_2 + d)$, which implies $x_1 = x_2$. Or use the fact that $f(x)$ is 1-to-1, since its graph is a hyperbola with horizontal and vertical asymptotes.
16	C	
17	D	$(5 \cdot 4)/(10 \cdot 9) = 2/9$.
18	A	$a/(1 - r) = 18$, $ar = 4$, $\therefore 18 = ar/(r - r^2) = 4/(r - r^2)$, $r - r^2 = 4/18$.

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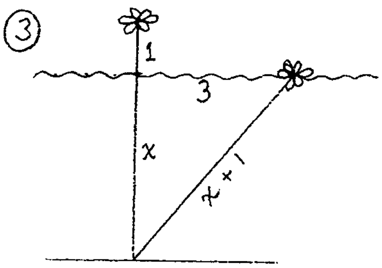

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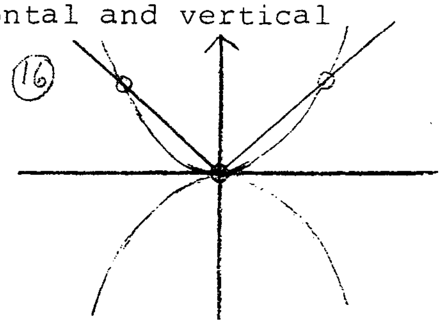

8 E $7^9 + 4 = (2 + 5)^9 + 4 = 2^9 + 9 \cdot 2^8 \cdot 5 + \cdots + 5^9 + 4$

$= 2^9 + 5(*) + 4 = 516 + 5(*) = 1 + 5 \cdot 103 + 5(*)$.

$\therefore$ rem $= 1$. Or use congruences mod 5.

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12 D The 3rd equation $= 3 \cdot (\text{first}) - 4 \cdot (\text{second})$. Or observe determinant $= 0$, ($\therefore$ C or D), and find one particular solution, such as $(0, 1/5, 1/5)$.

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16 C

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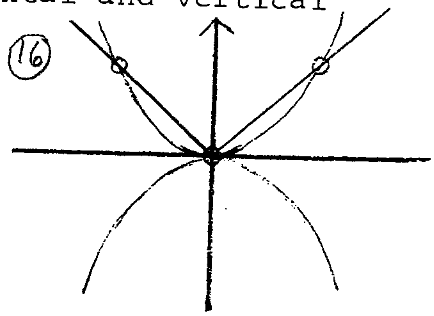

17 D $(5 \cdot 4)/(10 \cdot 9) = 2/9$.

18 A $a/(1 - r) = 18$, $ar = 4$, $\therefore 18 = ar/(r - r^2) = 4/(r - r^2)$, $r - r^2 = 4/18$.

19 C $x =$ edge. $6x^2/x^3 = 4$, $x = 3/2$, $\sqrt{3}x = 3\sqrt{3}/2$.

20 B $5x + \pi/3 = k\pi$.

21 E Rationalize by squaring. $\therefore 4x^2 - 5x + 1 = 0$, $x = 1, 1/4$. $x = 1/4$ is extraneous.

22 C $v_A/v_B = 4/3$, $v_B/v_C = 8/7$, $\therefore v_A/v_C = 32/21 = \text{dist}_A/\text{dist}_C$.

23 D (See sketch on the other side.)

No.	Ans.	Comments (Many problems can be solved in several ways.)
24	E	Determinant = 0, since (for example) subtracting the first row from the second and the third from the fourth gives two equal rows.
25	B	$\sqrt{4 - 2\sqrt{3}} + \sqrt{7 - 4\sqrt{3}} = \sqrt{(\sqrt{3} - 1)^2} + \sqrt{(2 - \sqrt{3})^2} = \sqrt{3} - 1 + 2 - \sqrt{3} = 1.$
26	B	$(x^2 + 2x)/(x^2 - 4x) = (x + 2)/(x - 4)$ if $x \neq 0$. This is negative if $x + 2 > 0$ and $x - 4 < 0$.
27	B	cos x uniformly larger than the others for $0 \le x \le \pi/4$.
28	D	$3 = 2^x$, $5 = 3^y$. $\therefore 10 = 2\cdot 5 = 3^{1/x}\cdot 5 = (5^{1/y})^{1/x}\cdot 5 = 5^{1/xy + 1} = 5^{(xy + 1)/xy}$. Or use $\log_a b = \log_c b/\log_c a$.
29	B	$C(10,3) + C(10,4) = \binom{10}{3} + \binom{10}{4} = 120 + 210 = 330.$
30	A	Value = (11/10) of original price paid, x. Selling price = (11/10) of value. $\therefore (11/10)[(11/10)x] - x = 10$. $\therefore x = 47.62$.
31	B	Let x = OM, y = MQ. From the three right triangles OTP, OMP, OMQ one has $OP^2 = 16 + 25 = 41$, $x^2 + (y + 3)^2 = 41$, $x^2 + y^2 = 16$. Solving, $x = 4\sqrt{5}/3$.
32	B	$a_2(x - x_1)(x - x_2) = a_2x^2 + a_1x + a_o$ yields $x_1 + x_2 = -a_1/a_2$, $x_1x_2 = a_o/a_2$. Use this twice to find coefficients of required polynomial.
33	A	Point P(3,2) is point of tangency. Tangent perpendicular to radius.
34	C	Radius = 4. Perimeter = $6(8 \tan 30^\circ) = 48\sqrt{3}/3 = 16\sqrt{3}$.
35	C	$6561 = 3^8$; $1 + 3 + 3^2 + \dots + 3^8 = (1 - 3^9)/(1 - 3) = 9841$, (geometric series).
36	B	$S_n = (1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \dots + (1/n - 1/n+1)$ $= 1 + (-1/2 + 1/2) + (-1/3 + 1/3) + \dots + (-1/n + 1/n) - 1/n+1$ $= 1 - 1/n+1 \to 1$.
37	C	$\theta = 30^\circ$. $\therefore$ measure of $\angle QOP = 120^\circ$, arc QP = (120/360)·(circumference of circle) = $200\pi/3$.
38	A	$\cos^2(\theta/2) = (1 + \cos\theta)/2 = (1 \pm \sqrt{1 - \sin^2\theta})/2 = 9/10$ or $1/10$; $\cos(\theta/2) = \pm 3\sqrt{10}/10$ or $\pm\sqrt{10}/10$.
39	D	$\cos A = (3^2 + 4^2 - 37)/(2\cdot 3\cdot 4) = -1/2$, (law of cosines). A obtuse. $\therefore \sin A =$

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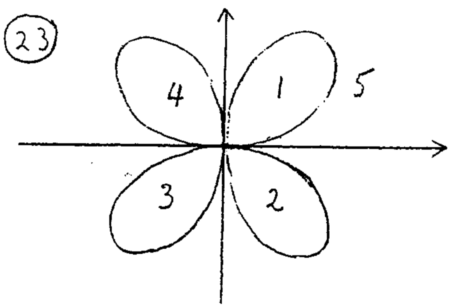

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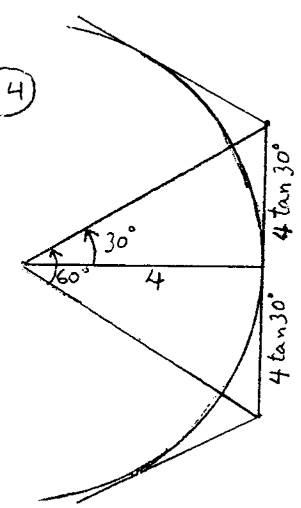

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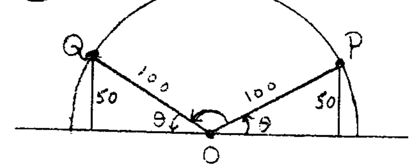

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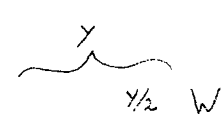

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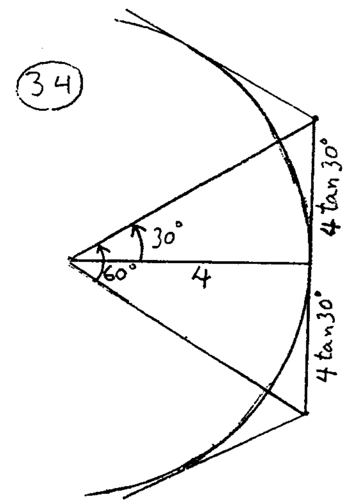

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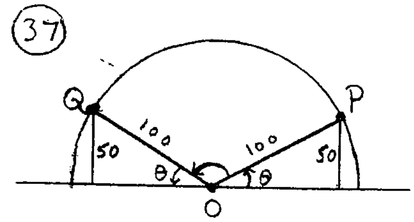

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39 D $\cos A = (3^2 + 4^2 - 37)/(2\cdot 3\cdot 4) = -1/2$, (law of cosines). A obtuse. $\therefore \sin A = +\sqrt{1 - (1/2)^2} = +\sqrt{3}/2$.

40 B $A = 2r\cdot(x + y)/2 = r(x + y)$, $\therefore x + y = A/r$. Since $2\alpha + 2\beta = 180^\circ$, $\alpha + \beta = 90^\circ$, and $\therefore$ $\triangle$'s OUV, OVW are similar. Thus $(x/2)/r = r/(y/2)$, $xy = 4r^2$. It follows that x and y satisfy $t^2 - (A/r)t + (4r^2) = 0$, which has solution B.