SIXTEENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America, Michigan Colleges and Universities, Professional Organizations, Industries, and Foundations.

PART 1

October 18, 1972

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Check to insure that your six-digit student number has been recorded correctly. Do not make calculations on the answer sheet.

2. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

3. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

4. The first 20 problems of this examination are intended to sample many of the topics in the secondary mathematics curriculum. You may be unfamiliar with some of these topics and quite possibly will find a number of problems which are easier for you distributed throughout the last twenty items. Usually a score of about 20 or more will allow you to become a finalist and write the second exam.

5. In each of the questions five different possible responses are proposed. In many cases the fifth alternative is listed "E" none of these. In such cases if you believe none of the first four alternatives to be correct, mark E.

6. Your score on the test will be the number correct. You are advised to guess an answer in those cases where you cannot determine the right answer or are able to eliminate some of the alternatives as impossible.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. If you have questions concerning the instructions ask them now.

16th ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

1. If a square of side x and a circle of radius 4 have the same area, then the ratio of the perimeter of the square to the circumference of the circle is

   A) $2/\sqrt{\pi}$ B) $2/\pi$ C) 1 D) $2/\pi^{3/2}$ E) none of these

2. The negation of the statement "All the theorems of mathematics are tautologies" is

   A) All tautologies are theorems of mathematics.

   B) All the theorems of mathematics are not tautologies.

   C) There is a theorem of mathematics which is not a tautology.

   D) Some of the theorems of mathematics are tautologies.

   E) none of these

3. A water lily with rigid stem extends one foot above the surface of the water. When pulled over, it disappears under the water at a distance three feet from where the stem originally entered the water. How deep is the lake?

   A) 4 ft. B) $\sqrt{10}$ ft. C) $\sqrt{3\pi}$ ft. D) $\sqrt{10}\,\pi$ ft. E) none of these

4. You are given the number 1972 expressed in the usual system of base 10. Express this number in the system of base 8.

   A) 1972 B) 3214 C) 3464 D) 3664 E) none of these

5. A circular grindstone has a diameter of 3 feet. During use the outer edge is worn away. What will be the diameter of the stone when the stone is half gone?

   A) 1.5 ft B) $3\pi/2$ ft. C) $3\pi\sqrt{2}/2$ ft. D) $3\sqrt{2}/2$ ft.

   E) none of these

6. A mother weighs 100 lbs. more than the combined weights of her baby and her dog. Together all three weigh 172 lbs. Her dog weighs forty percent of the weight of the baby. The baby weighs

A) 32 lbs. B) 28 4/5 lbs. C) 25 5/7 lbs.

D) 22 1/2 lbs. E) none of these

7. Consider the following statements about prime numbers.

(i) The sum of two prime numbers may be a prime number.

(ii) If a prime is not a factor of either of two integers then it is not a factor of their product.

(iii) Every non-prime integer greater than 1 is the product of primes.

A) Only (iii) is true B) Only (i) and (ii) are true

C) Only (i) and (iii) are true D) Only (ii) and (iii) are true

E) All three are true

8. If $7^9 + 4$ is divided by 5, the remainder is

A) 0 B) 4 C) 3 D) 2 E) 1

9. The polynomial $x^4 - 7x^2 + 12$ has

A) No real roots B) Exactly two real roots

C) Exactly three real roots D) Exactly four rational roots

E) none of these

10. If $x^4 - x^3 + 3x^2 + 2x + 4$ is divided by $x + 1$, the remainder obtained is

A) 7 B) 0 C) -3 D) 9 E) none of these

11. Consider the following three statements about the equation $ax^2 + bx + c = 0$, where a, b, and c are integers.

(i) The equation always has the same number of positive as negative roots.

(ii) The equation always has an even number of irrational roots.

(iii) The equation always has an even number of complex roots.

A) Only (i) is true B) Only (ii) is true

C) Only (iii) is true D) Only (ii) and (iii) are true

E) All of the three statements are true.

12. The system of equations

$$3x - y + z = 0$$

$$2x + y + 4z = 1$$

$$x - 7y - 13z = -4$$

A) has exactly one solution B) has exactly three solutions

C) has no solutions D) has infinitely many solutions

E) none of these

13. The sum of the solutions of the equation

$|2x + 1| + |x - 4| = 12$ is

A) 2 B) 5 C) 2/3 D) 23/3 E) none of these

14. If $0 < a < b$, $c < d < 0$, then necessarily

A) $ac < bd$ B) $ad < bc$ C) $bc < ad$

D) $ad < bc$ E) $bd < ac$

15. A function $f(x)$ is defined for $x \neq -\frac{d}{c}$ by $f(x) = \frac{ax+b}{cx+d}$ where $ad - bc \neq 0$ and $c \neq 0$. Then:

A) $f(x_2) = f(x_1)$ for $x_2 = x_1 + ad - bc$

B) $f(x_2) = f(x_1)$ only if $x_2 = x_1$

C) $f(x_2) = f(x_1)$ for $x_2 = x_1 + c$

D) $f(x_2) = f(x_1)$ for a non-zero finite number of $x_2 \neq x_1$

E) $f(x_2) = f(x_1)$ only if $x_1 = x_2 = 0$

16. The graphs of $y = |x|$ and $x^2 = |y|$ have

A) exactly one point in common

B) exactly 2 points in common

C) exactly 3 points in common

D) exactly 4 points in common

E) none of these

17. A box contains ten balls numbered from 1 to 10 inclusive. One ball is drawn and, without replacing it, a second drawing is made. The probability that both balls have even numbers is:

A) 1/5 B) 1/3 C) 1/4 D) 2/9 E) none of these

18. If the sum of an infinite geometric progression is 18 and the second term is 4, then the common ratio r of successive terms is

A) 1/3 or 2/3 B) 1/2 or 3/4 C) 3/5 or 1/5

D) 1/3 only E) none of these

19. The ratio of the surface area of a solid cube to its volume is 4 to 1. The diagonal of the cube is

A) 3/2 B) 2/3 C) $3\sqrt{3}/2$ D) $3\sqrt{2}$

E) none of these

20. If $\sin\left(5x + \frac{\pi}{3}\right) = 0$, then

A) $x = \frac{\pi}{15}$ B) $x = \frac{(3k-1)\pi}{15}$ where k is an integer. C) $x = 12$

D) $x = \frac{(3k-2)\pi}{15}$ where k is an integer. E) none of these

21. The solution(s) of the equation $\sqrt{5x-1} = \sqrt{x} + 1$ is(are)

A) 1/2 B) $\frac{5 \pm \sqrt{41}}{8}$ C) there are no solutions

D) two numbers whose sum is 5/4 E) none of these

22. In a race of two miles, A beats B by 1/2 mile.

In a race of two miles, B beats C by 1/4 mile.

In a race of two miles, by how much does A beat C? (Assuming A, B, and C run at constant speeds)

A) 5/8 miles B) 21/32 miles C) 11/16 miles

D) 3/4 miles E) none of these

23. The graph of $r = 5 \sin 2\theta$ in polar coordinates divides the plane into what number of regions?

A) 2 B) 3 C) 4 D) 5 E) none of these

24. The determinant $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix}$ is

A) 125 B) -15 C) -78 D) 108 E) none of these

25. $\sqrt{4-2\sqrt{3}} + \sqrt{7-4\sqrt{3}} =$

A) $\sqrt{11-6\sqrt{3}}$ B) 1 C) $11 - 6\sqrt{3} + 2\sqrt{(4-2\sqrt{3})(7-4\sqrt{3}}$

D) $3 - 2\sqrt{3}$ E) none of these

26. $\frac{x^2+2x}{x^2-4x} < 0$ if and only if

A) $-2 < x < 4$ B) $4 > x > -2$ and $x \neq 0$

C) $x > 4$ or $x < -2$ D) $x < -2$ E) none of these

27. Consider the area enclosed by the vertical lines $x = 0$ and $x = \pi/4$, which lies above the x-axis and below the function $f(x)$. For which of the following $f(x)$ is this area the greatest?

A) $f(x) = \sin x$ B) $f(x) = \cos x$ C) $f(x) = \sin^2 x$

D) $f(x) = \cos^2 x$ E) $f(x) = 1/2$

28. Suppose $\log_2 3 = x$ and $\log_3 5 = y$. Then $\log_5 10$ expressed in terms of x and y is equal to:

A) $\frac{x+1}{xy}$ B) $\frac{xy}{xy+1}$ C) $\frac{y+1}{xy}$ D) $\frac{xy+1}{xy}$ E) $\frac{xy}{y+1}$

29. Suppose a set has 10 elements. The number of subsets which have either 3 or 4 elements is

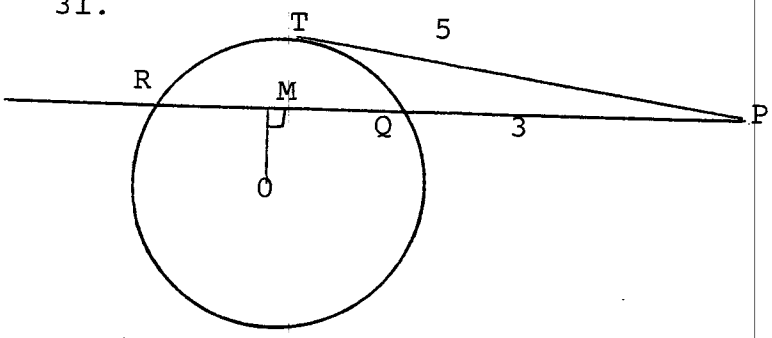
A) 195 B) 330 C) 55 D) 210 E) none of these

30. A dishonest merchant has a set of scales which balances when 11 lbs. are in one pan and 10 lbs. are in the other. His advertised buying and selling prices of grain are identical, but he cheats the farmer in buying, and the buyer in selling. He buys a fixed amount, and then resells it, and makes \$10. How much did he pay for the original purchase (to the nearest cent)?

A) \$47.62 B) \$50.00 C) \$52.36

D) \$100 E) none of these

31.



$\overleftrightarrow{PT}$ is tangent to the circle centered at 0 of radius 4, with T the point of tangency.
PT = 5, PQ = 3
$\overleftrightarrow{OM} \perp \overleftrightarrow{RQ}$

A) OM = 2 B) OM = $4\sqrt{5}/3$ C) OM = 3

D) OM = 5/2 E) OM is not determined by this data

32. If r_1 and r_2 are nonzero roots of the quadratic function $3x^2 + ax + b$, then $\frac{1}{r_1^2}$ and $\frac{1}{r_2^2}$ are roots of the quadratic function

A) $b^2x^2 + 9a^2 + 1$ B) $b^2x^2 + (6b-a^2)x + 9$

C) $bx^2 + ax + 3$ D) $3x^2 + b^2x + a^2$ E) none of these

33. If the line $y - 2 = m(x-3)$ is to be a tangent line to the circle $(x-5)^2 + (y - \frac{1}{2})^2 = \frac{25}{4}$ then m must be equal to:

A) 4/3 B) 2/3 or -3 C) 3/4 D) - 4/3

E) none of these

34. The perimeter of a regular hexagon circumscribed about a circle of circumference 8π is

A) 24 B) 27.2 C) $16\sqrt{3}$ D) $24\sqrt{3}$ E) none of these

35. The sum of the divisors of 6561 is

A) 9661 B) 9422 C) 9841 D) 9901

E) none of these

36. If $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n\cdot(n+1)}$ then the larger the value of n, the closer S_n approximates

A) .95 B) 1 C) $\pi/4$ D) .9 E) none of these

37. A large gateway arch is in the form of a semicircle of radius 100 feet. Two lights are placed on the arch at a height of 50 feet above the ends of the arch (one on each side). What is the length of the arch between the lights?

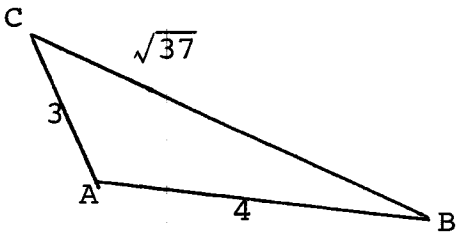
A) 200 feet B) 50π feet C) $200\pi/3$ feet

D) $50\pi\sqrt{3/2}$ feet E) none of these

38. If $\sin\theta = -3/5$, then one possible value of $\cos(\theta/2)$ is

A) $-3\sqrt{10}/10$ B) $-3/10$ C) $\sqrt{3}/10$ D) $3/10$

E) none of these

39. 

In Δ ABC

A) $\sin A = -\sqrt{3}/2$

B) $\tan A = -\sqrt{3}/22$

C) $\sin A = 1/2$

D) $\sin A = \sqrt{3}/2$ E) none of these

40. An isosceles trapezoid of area A is circumscribed about a circle. If the radius of the circle is r then the lengths of the parallel sides are:

A) $\frac{2A \pm \sqrt{A^2-8r^4}}{4r}$ B) $\frac{A \pm \sqrt{A^2-16r^4}}{2r}$ C) $\frac{A \pm \sqrt{A^2-16r^2}}{2r}$

D) $\frac{A \pm \sqrt{A^2-8r^4}}{2r}$ E) none of these

The following Michigan companies have made financial contributions to the Michigan Mathematics Prize Competition.

Hammond Machinery Builders, Inc., Kalamazoo

Michigan Bell Telephone Company, Detroit

Agencies submitting contributions to this competition after the printing of this examination will be recognized at the Annual Awards Banquet sponsored by Michigan Bell Telephone.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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