SOLUTIONS II

FIFTEENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

1. If n is an integer, one of the following relations holds:

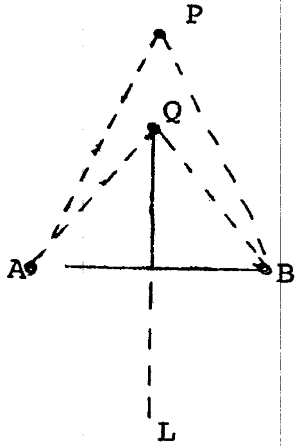
$$n \equiv 0,1,2,3,4,5,6 \pmod 7$$

Then $n^2 \equiv 0,1,4,2,2,4,1 \pmod 7$

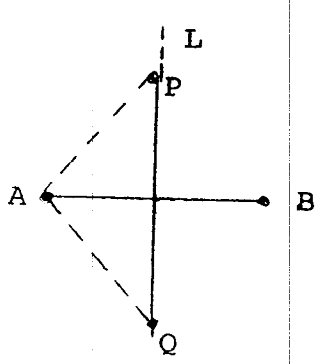
and $n^2+1 \equiv 1,2,5,3,3,5,2$ so. $n^2+1 \not\equiv 0 \pmod 7$

2. If we subtract the first equation from the second, and the second from the third, we obtain $(y-x)u=1$ and $(z-y)u=1$, where $u = x+y+z$. Then $y-x = z-y$, $x+z = 2y$, $u = 2y+y = 3y$. Now $x = y - \frac{1}{u} = y - \frac{1}{3y}$, $z = y + \frac{1}{u} = y + \frac{1}{3y}$. Also $y^3 = 2+xz = 2y+y^2 - \frac{1}{9y^2}$, or $y^2 = \frac{1}{18}$, $y = \pm\sqrt{2}/6$. Then $x = \frac{-5}{6}\sqrt{2}$, $y = \frac{\sqrt{2}}{6}$, $z = \frac{7\sqrt{2}}{6}$, and a second solution is $x = \frac{5}{6}\sqrt{2}$, $y = \frac{-\sqrt{2}}{6}$, $z = \frac{-7\sqrt{2}}{6}$.

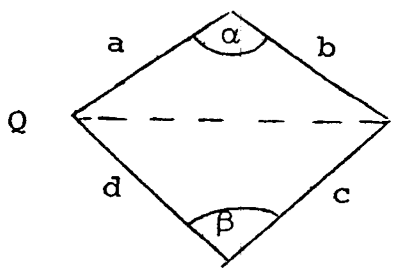
3. The vertex angle of a triangle with long legs cannot by obtuse, since the greatest side is opposite the greatest angle. Let us show n=3. Suppose there are n points in the plane with the property mentioned in the problem, so $n \geqq 3$. Let AB be the shortest distance between any two of the n points. Then all other points in the set lie on the perpendicular bisector L of AB.



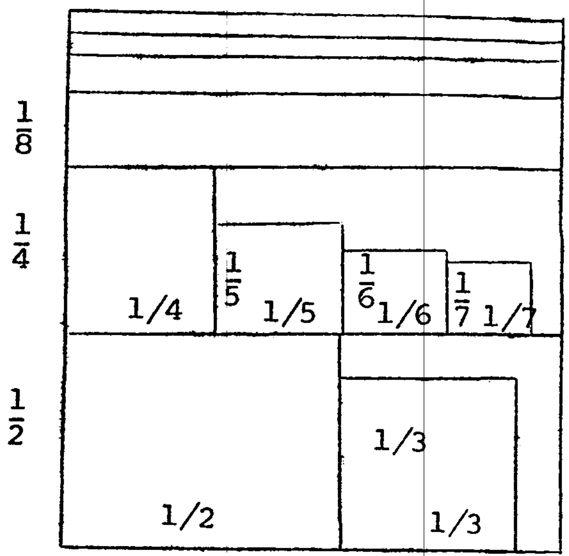
<u>Case 1</u>. P and Q lie on L and on one side of AB. Let Q be inside ΔPAB. Join Q to A on B. Clearly, one of the angles at Q is obtuse, a contradiction. ($\angle AQP$ for example).



<u>Case 2</u>. P and Q on L lie on opposite sides of AB. Since $PA \geqq AB$ and $QA \geqq AB$, then $\angle PAQ$ is obtuse, (in fact is at least 120°), a contradiction. $\therefore$ <u>n=3</u>



4. Since Q is inscriptable, $\alpha+\beta = 180°$, and for future use we note that $\sin\beta = \sin\alpha$ and $\cos\beta = -\cos\alpha$. Since Q is circumscriptable, $a+c = b+d$, a fact easily proved using equal tangents to a circle from an outside point. Now $A = \frac{1}{2} ab\sin\alpha + \frac{1}{2} cd\sin\beta = \frac{1}{2}\sin\alpha\,(ab+cd)$ and $4A^2 = \sin^2\alpha\,(ab+cd)^2 = (1-\cos^2\alpha)(ab+cd)^2$. By the Law of Cosines, $a^2+b^2-2ab\cos\alpha = c^2+d^2-2cd\cos\beta = c^2+d^2+2cd\cos\alpha$. Then $2\cos\alpha(ab+cd) = a^2+b^2-c^2-d^2$. Since $a-b = d-c$, then by squaring, $a^2+b^2-c^2-d^2 = 2(ab-cd)$, so $\cos\alpha = \frac{ab-cd}{ab+cd}$. Inserting this value of $\cos\alpha$ into the formula for $4A^2$, $4A^2 = (ab+cd)^2 - (ab-cd)^2 = 4abcd$, so $A = \sqrt{abcd}$.



5. Divide the unit square into layers, as shown. Note that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is a geometric progression whose sum to infinity is $\frac{a}{1-r} = \frac{1/2}{1 - 1/2} = 1$. In first layer, place the squares of sides $\frac{1}{2}$ and $\frac{1}{3}$ $(\frac{1}{2} + \frac{1}{3} < 1)$. On the second layer place the four squares of sides $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ Note: Sum $< 4(\frac{1}{4}) = 1$. In nth layer insert 2^n squares of sides $\frac{1}{2^n}, \frac{1}{2^n+1}, \frac{1}{2^{n+1}-1}$. Sum of sides $< 2^n(\frac{1}{2^n}) = 1$.

Thus there is no overlap in area, and all small squares have been accounted for.

[Note: By Fourier Series Methods, $\sum_{n=2}^{\infty} \frac{1}{n^2} = \underbrace{\frac{\pi^2}{6} - 1}_{\text{about } .645} < 1.$]

So there is room