FIFTEENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America, Michigan Colleges and Universities, Professional Organizations, and Industries.

PART 1

October 20, 1971

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

2. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

3. Your score on the test will be the number right. You are advised to guess an answer in those cases where you cannot determine the right answer but are able to eliminate some of the alternatives as impossible.

4. Usually a score of 20 or more will allow you to become a finalist and write the second exam. To improve your score, be careful to complete all problems which you can do successfully before working on the other problems. Several questions, distributed throughout the test, can be answered by using only the first two years of high school mathematics.

5. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Do not make calculations on the answer sheet.

6. In each of the questions five different possible responses are proposed. In many cases the fifth alternative is listed "E none of the above." In such cases if you believe none of the first four alternatives to be correct, mark E.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. If you have questions concerning the instructions ask them now.

15th ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

1. The numeral 2222222 is in base 3; the same number in base 10 would have the numeral

   (a) 4364 (b) 2186 (c) 2183 (d) 728

   (e) None of the above

2. If there are n points in the plane no three of which are colinear and they determine 351 lines, n is equal to

   (a) 54 (b) 27 (c) 26 (d) 13

   (e) None of the above

3. The number of subsets of the set $\{a,b,c,d,e,f\}$ is

   (a) 64 (b) 63 (c) 36 (d) 32 (e) 6

4. A right circular cone has height h and base radius r; the area of the cone (excluding area of the base) is

   (a) $\pi r\sqrt{r^2 + h^2}$ (b) $\pi(r^2 + h^2)$ (c) $\pi h\sqrt{r^2 + h^2}$

   (d) $\pi r h$ (e) None of the above

5. The number of integers between 1 and 250 (inclusive) that are not divisible by 2 nor by 7 but are divisible by 5 is

   (a) 21 (b) 20 (c) 19 (d) 18

   (e) None of the above

6. The value of $\dfrac{a^n}{2na^n - 2nx} + \dfrac{b^n}{2nb^n - 2nx}$ for $x = \dfrac{a^n + b^n}{2}$

and $0 < a < b$ is

(a) $\frac{1}{n}[(\frac{a}{b})^n - (\frac{b}{a})^n]$ (b) $b^n - a^n$ (c) $\frac{1}{2n}$ (d) $\frac{1}{n}$

(e) $\dfrac{2}{n(a^n - b^n)}$

7. Given $\log_{10}2 = .3010$ and $\log_{10}3 = .4771$; then, correct to 2 decimal places, the set of all x which satisfy the equation

$$16^x - 8(4^x) + 12 = 0$$

is

(a) $\pm .50$, 1.29 (b) $.50$ (c) $.50$, 1.27

(d) $.50$, 1.29 (e) $.50$, 1.31

8. If $P(x) = ax^3 + bx^2 + cx + d$, where a, b, and d are non-zero real numbers and c is a non-zero strictly complex (i.e. not real) number, then the number of real roots of the equation $P(x) = 0$ is

(a) 3 (b) 2 (c) 1 (d) 0

(e) None of the above

9. Let $P(x) = x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be an integer polynomial (i.e. the a_i's are integers). Then 3 is not a root of $P(x) = 0$ for any set of a_i's, with

(a) $a_0 = 15$ (b) $a_1 = 4$, $a_0 = 6$

(c) $a_2 = 1$, $a_1 = 2$, $a_0 = 12$

(d) $a_3 = 2$, $a_2 = 3$, $a_1 = 4$, $a_0 = 15$

(e) In each of the above cases, one can find a_i's integral such that 3 is a root of $P(x) = 0$.

10. On a circle, 13 distinct points are placed at random and all possible line segments (chords) joining them in pairs are drawn. If no three of these chords are concurrent, then the number of intersections of these chords (not counting the original 13 on the circumference) is

(a) 3003 (b) 2080 (c) 715 (d) 160 (e) 78

11. A geometric progression with first term 3 is such that the sum of the first three terms is 9. The common ratio is

(a) −2 or 1 (b) 2 (c) 1 (d) −1

(e) None of the above

12. If $2 \sin \theta = \sin 2\theta$,

(a) then $\theta = k\pi$ (k an integer)

(b) then the equation is true for all θ

(c) then $\theta = \frac{\pi}{6}$ (d) has no solution (e) $\theta = \frac{\pi}{4}$

13. A $3 \times 3 \times 3$ cube is painted red. The cube is cut into 27 one by one by one cubes. The number of the small cubes which have exactly 2 red sides is

(a) 4 (b) 8 (c) 12 (d) 16

(e) None of the above

14. There is a natural number n such that $\sqrt[n]{7 - 2\sqrt{10}} =$

(a) $\sqrt{7} + \sqrt[3]{5}$ (b) $\sqrt{5} - \sqrt{2}$ (c) $\sqrt[6]{10} - 7\sqrt[3]{2}$

(d) $\sqrt[3]{7} - \sqrt[3]{10}$ (e) There is no n such that any of the above hold.

15. Which of the following functions takes on negative values for negative values of x in its domain, and positive values for positive x?

(a) $\frac{1}{x} - \frac{1}{x+1}$ (b) $(-x)^2 + x$ (c) $x - |\sin x|$

(d) $x + \cos x$ (e) None of the above

16. A train 1 mile long is moving at a speed of 50 m.p.h. Two men, starting from opposite ends of the train, walk toward each other. If the man starting at the rear of the train walks at a speed of 2 m.p.h. and the man starting at the front of the train walks at a speed of 3 m.p.h., how long does it take before they meet?

(a) 1/99 hours (b) 12 minutes (c) 10 minutes

(d) 5 minutes (e) None of the above

17. A regular hexagon is inscribed in one circle and circumscribed about another. The ratio of the areas of the two circles (larger to smaller) is

(a) 2 : 1 (b) $2 : \sqrt{3}$ (c) 4 : 3

(d) Dependent upon the size of the hexagon

(e) None of the above

18. If the total area of a cone (base included) equals 54π in^2 and the length of the lateral side $\ell = 15$ inches, then the height of the cone is:

(a) $8\sqrt{2}$ (b) $6\sqrt{6}$ (c) 16 (d) $9\sqrt{3}$

(e) None of the above

19. Suppose $\log_8 3 = x$ and $\log_3 5 = y$. Then $\log_{10} 6$ expressed in terms of x and y is equal to:

(a) $\dfrac{2x + 1}{2xy + 1}$ (b) $\dfrac{3x - 1}{3xy - 1}$ (c) $\dfrac{3x + 1}{3xy + 1}$

(d) $\dfrac{1 - 3x}{1 - 3xy}$ (e) None of the above

20. For each positive integer n let $f_{n+1}(x) = f_1(f_n(x))$, where $f_1(x) = 2x - x^2$. Then for all n,

(a) $f_n(x) = 2(2x - x^2)^n - (2x - x^2)^{2n}$

(b) $f_n(x) = (2x - x^2)^n$ (c) $f_n(x) = 1 - (1 - x)^{2^n}$

(d) $f_n(x) = (1 - (1 - x)^2)^n$ (e) None of the above

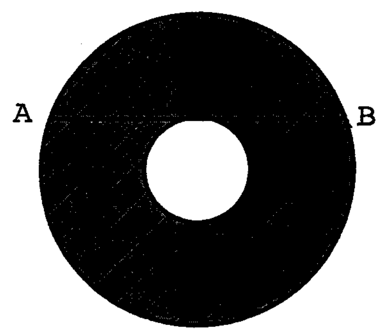
21. The equation $|x - 1| + |x + 1| + |x - 2| = 12$ has:

(a) no solution (b) only one solution

(c) only two solutions (d) only three solutions

(e) None of the above

22. In the diagram below, line segment AB is 2 inches long and tangent to the smaller of the two concentric circles. The area of the shaded region is:



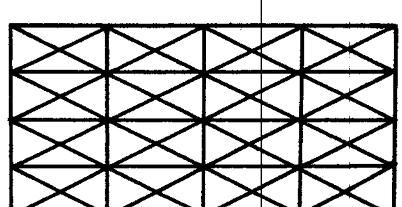
(a) π inches^2

(b) $\frac{\pi^2}{4}$ inches^2

(c) $\frac{\sqrt{3}\pi}{2}$ inches^2 (d) $\frac{3}{\sqrt{\pi}}$ inches^2

(e) None of the above

23. The pattern below cannot be traced in one continuous path, tracing a given line exactly once, without lifting pencil from paper.



The fewest number of times you must lift your pencil from the paper to continuously trace this pattern (with no retracing of any line) is

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

24. In how many ways can 12 balls of different colors be put into 3 separate trays, if the first tray must contain 3 balls, the second 4 balls, and the third 5 balls?

(a) 6,056 (b) 8,172 (c) 18,032 (d) 27,720
(e) 64,920

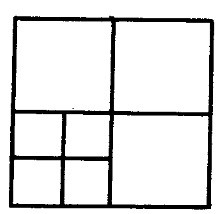
25. The number of zeros with which the number

$$500! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 499 \cdot 500$$

terminates is exactly

(a) 72 (b) 124 (c) 156 (d) 187
(e) None of the above

26. Below is a subdivision of a square into seven nonoverlapping squares



For $n \leq 10$, a square can be subdivided into n non-overlapping squares if and only if

(a) $n \neq 2$ (b) $n \neq 2, 3$ (c) $n = 4, 6, 7$

(d) n is a prime number (e) None of the above

27. The entries in a determinant of order 4 are filled in at random with 0's and 1's. The probability that the determinant is odd is

    (a) 1/2 (b) 105/1024 (c) 315/1024

    (d) 3/8 (e) None of the above

28. At what time between 3 and 4 o'clock do the minute and hour hands of a clock coincide?

    (a) 16 1/4 minutes after 3 o'clock

    (b) 16 4/11 minutes after 3 o'clock

    (c) 13 3/11 minutes after 3 o'clock

    (d) 16 5/12 minutes after 3 o'clock

    (e) None of the above

29. A fly is crawling on the surface of a cubical block of side length 1 foot. The shortest distance he must travel to crawl from a vertex to the opposite vertex (the vertex farthest from the initial vertex) is

    (a) $\sqrt{2}$ feet (b) $\sqrt{3}$ feet (c) $\sqrt{5}$ feet

    (d) $\sqrt{10}$ feet (e) 3 feet

30. Man A can do a job in 3 hours. Woman B can do the same job in 2 hours. How long does it take A and B working together to do the same job.

(a) 7/6 hours (b) 3/2 hours (c) 1 hour

(d) 6/5 hours (e) None of the above

31. What is the volume of a cube inscribed in a sphere of radius 1.

(a) $\frac{8}{3\sqrt{3}}$ (b) $\frac{1}{2}$ (c) $\frac{8}{9\sqrt{3}}$ (d) $\frac{8}{9}$

(e) None of the above

32. A triangle ABC is inscribed in a circle of radius 18 inches. If the angles at the vertices A, B, and C are in the ratio 2 : 3 : 4 then the arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CA}$ of the circle are respectively:

(a) 21π, 7π, 8π (b) 18π, 6π, 12π

(c) 16π, 8π, 12π (d) 7π, 8π, 12π

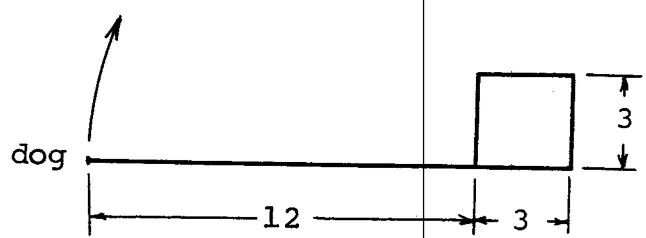
(e) None of the above

33. Given $\tan\alpha$, there are two possible values for $\tan\frac{\alpha}{2}$; their product is:

(a) $\frac{2}{\tan\alpha}$ (b) $\frac{2\tan\alpha}{1+\tan^2\alpha}$ (c) -1

(d) $\frac{2\tan\alpha}{1-\tan^2\alpha}$ (e) None of the above

34. A dog is tied to a 3×3 -yard square log cabin with a 12 yard long rope



If he starts in the position shown and runs clockwise about the cabin, always keeping the rope taut, how many yards can he run?

(a) 7.5π yards (b) 15π yards (c) 24π yards

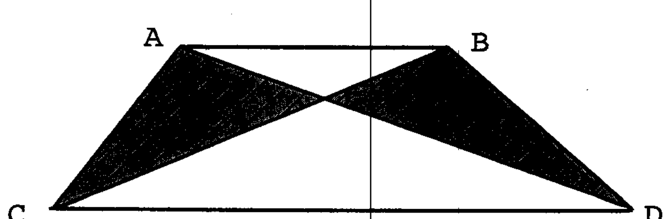
(d) 30π yards (e) None of the above

35. The largest number r such that 2 circles, each of radius r, can be placed without overlap in a square of side length 2, is

(a) $\sqrt{2}/2$ (b) $\dfrac{1+\sqrt{2}}{2\sqrt{2}}$ (c) $\dfrac{1+\sqrt{2}}{\pi}$

(d) $\dfrac{\sqrt{2}}{1+\sqrt{2}}$ (e) None of the above

36. In the figure below, AB is parallel to CD, but AC is <u>not</u> parallel to BD.



The fraction of the total area of trapezoid ABCD represented by the shaded area above is

(a) 1/2 (b) less than 1/2 (c) more than 1/2

(d) may be any of (a), (b) or (c) above, depending on the trapezoid given.

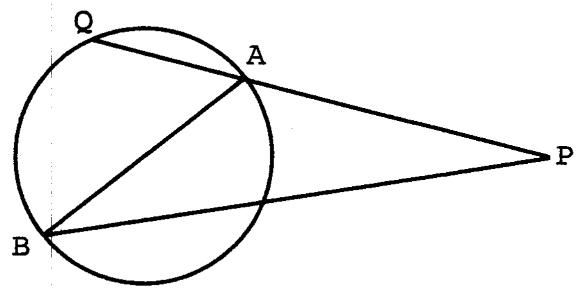
(e) None of the above

37. In the figure, $\overline{AB}$ is a diameter of the circle. If $AB = 8$, $AQ = 4$ and $PQ = 12$, then $PB =$

(a) $8\sqrt{3}$ (b) 12

(c) $12\sqrt{3}$ (d) $4\sqrt{6}$

(e) None of the above



38. The real solutions of the equation $\log_2(6x + 5) + \log_2 x = 2$ are

(a) $1/2$, $-4/3$ (b) $1/2$ (c) $-4/3$ (d) $-3/7$

(e) $-1/7$

39. The centers of two circles of radii 7 inches and 1 inch are 10 inches apart. Then the lengths of the common tangents are:

(a) 6, 8 (b) 7, 9 (c) $5\sqrt{2}$, $7\sqrt{2}$

(d) $4\sqrt{3}$, $6\sqrt{3}$ (e) None of the above

40. The number of elements in the set of ordered n-tuples $\{(P_1, P_2, \ldots, P_n) : P_1, P_2, \ldots, P_n$ are nonnegative integers $P_1 + P_2 + \ldots + P_n \leq m\}$ is

(a) n^m (b) $\frac{(m + n)!}{m!\, n!}$

(c) The sum of the prime numbers $\leq m\, n$

(d) $\frac{n^m (m + n)!}{m!}$ (e) None of the above

The following Michigan companies and professional organizations have made contributions to the scholarship fund for this competition.

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