FOURTEENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

REPORT OF THE GRADING COMMITTEE

Solutions to Problems of Part II

Taken December 9, 1970

The gold medalist, David Garlock, of Southfield high school wrote a perfect paper on Part II. His solutions were so elegantly conceived and presented that the committee thought that it could do no better than present his verbatim arguments as a model.

A few comments concerning grading guide lines.

Problems 1 and 2: Relatively little credit was given for checking a few low order cases.

Problem 3: It is easy to strip Garlock's proof of the language of modular arithmetic and present a completely elementary ad hoc argument.

Problem 4: With the broad hint included this became a relatively easy problem so high standards for completeness and clarity of argument were set. For example some students calculated the length of the base of the isoceles triangle associated with the smallest angle of the triangle (case 2 in Garlock's solution). This leads to the right answer but some justification that the small angle must produce the small base segment was required to obtain major credit.

Of course the way to avoid such an argument is to compute the three possible candidates and compare the results.

Problem 5: There was much talking and manipulating on this problem but very few efforts got off the ground.

Generally the grading committee felt that the examination was quite reasonable. It felt that well grounded high school juniors and seniors with some degree of mathematical sophistication could quite reasonably be expected to get major credit on from two to four problems. We judged problem 5 to be quite hard for the time allotted but the others were not too difficult.

The committee was astounded by the Garlock performance but was disappointed at the small number of contestants scoring above five points on Part II.

Those Part II papers with scores above 5 were regraded and relatively few grading errors were noted. However, a few contestants who eventually reached the top 100 did so primarily by virtue of their high score on Part I and these escaped our regrading. The committee belatedly checked these few papers and found only one error but unfortunately it resulted in a significant change in the score of the Part II paper.

Accordingly, we offer our apologies to Daniel O'Dowd of Rochester Adams High School for our failure to include him in the original list of prize winners at the banquet.

L. M. Kelly
For the Grading Committee

1. Show that the $n \times n$ determinant

$$\begin{vmatrix} 1+x & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & 1+x & 1 & \cdots & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1+x \end{vmatrix} = M$$

has the value zero when $x = -n$.

By a well-known law of matrix manipulation, the determinant is not changed when one column is added term for term to another column. Thus we can add the second, then third, then the fourth, and so on until the $n+n$ column to the first column. Thus

$$M = \begin{vmatrix} n+x & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ n+x & 1+x & 1+x & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ n+x & 1 & 1 & 1+x & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ n+x & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1+x \end{vmatrix}$$

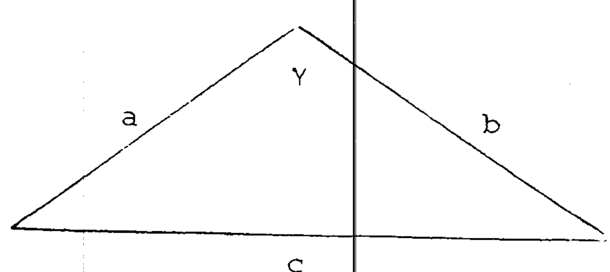
or

$$M = (n+x) \begin{vmatrix} 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & 1+x & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & 1 & \vdots & & & & & & \vdots \\ 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1+x \end{vmatrix}$$

Clearly $M = 0$ when $n+x = 0$ or when $x = -n$.

---

2. Let $c > a \geq b$ be the lengths of the sides of an obtuse triangle. Prove that $c^n = a^n + b^n$ for no positive integer n.



By the law of cosines, $c^2 = a^2 - 2ab\cos\gamma$. Since $\frac{\pi}{2} < \gamma < \pi$, $\cos\gamma < 0$ and therefore $-2ab\cos\gamma > 0$. Thus we have the inequality $c^2 > a^2 + b^2$. Now $c^n = c^{n-2}(c^2)$ and from the previous inequality $c^n > c^{n-2}(a^2 + b^2)$ or $c^n > c^{n-2}a^2 + c^{n-2}b^2$. Since we are given that $c > a$ and $c > b$, we know $c^k > a^k$ and $c^k > b^k$ for all $k > 0$. Therefore we can substitute a^{n-2} and b^{n-2} for c^{n-2} in the appropriate places and the inequality becomes

$$c^n > c^{n-2}a^2 + c^{n-2}b^2 > a^{n-2}a^2 + b^{n-2}b^2 = a^n + b^n$$

or simply $c^n > a^n + b^n$ for all $n > 2$. For $n = 2$ we have $c^2 > a^2 + b^2$ by the law of cosines and if $n = 1$ we have $c < a + b$ which is true in any triangle with sides a, b, and c.

3. Suppose that $p_1 = p_2^2 + p_3^2 + p_4^2$, where p_1, p_2, p_3, and p_4 are primes. Prove that at least one of p_2, p_3, p_4 is equal to 3.

Suppose $p_2 \neq 3$, $p_3 \neq 3$, $p_4 \neq 3$. Since these are all primes we have $p_2 \equiv \pm 1 \bmod 3$, $p_3 \equiv \pm 1 \bmod 3$, $p_4 \equiv \pm 1 \bmod 3$ and squaring we find:

$$p_2^2 \equiv 1 \bmod 3 \qquad p_3^2 \equiv 1 \bmod 3 \qquad p_4^2 \equiv 1 \bmod 3 \quad \text{and}$$

$$p_2^2 + p_3^2 + p_4^2 \equiv 0 \mod 3.$$

But $p_1 \equiv 0 \bmod 3 \Rightarrow p_1 = 4$ since all other numbers congruent to zero mod 3 have a factor of 3. But p_1 clearly does not equal 3 since the inequalities $p_2 \geq 2$, $p_3 \geq 2$, $p_4 \geq 2$ imply $p_2^2 + p_3^2 + p_4^2 \geq 12$. Thus we have a contradiction and our original assumption that none of p_2, p_3, and p_4 were 3 must be false.

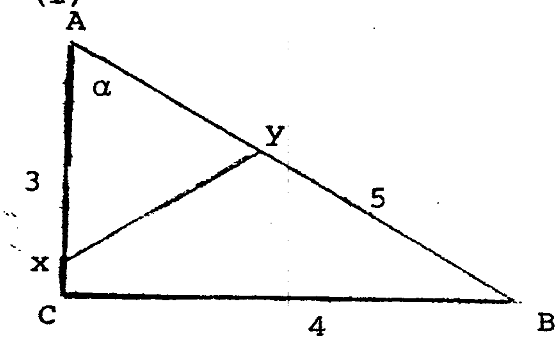
4. Suppose X and Y are points on the boundary of the triangular region ABC such that the segment XY divides the region into two parts of equal area. If XY is the shortest such segment and AB = 5, BC = 4, AC = 3 calculate the length of XY.

Hint: Of all triangles having the same area and same vertex angle the one with the shortest base is isosceles.

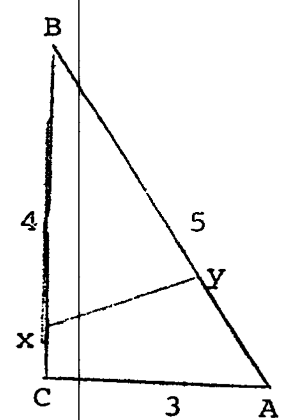
Clearly justify all claims.

There are essentially three possibilities for the position B of the segment.

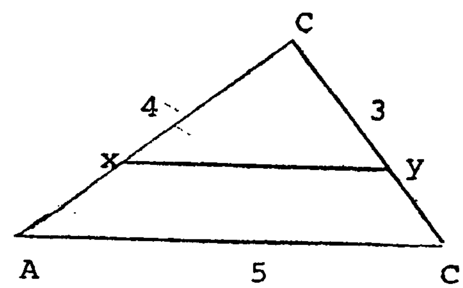
(1)



(2)



(3)



The area of the given triangle is $\frac{1}{2}\cdot 3\cdot 4 = \underline{6}$.

Case 1: Let $AX = S_1$. Since we wish to minimize XY, we can use the hint so $AY = S_1$. Let A_1 be area $\Delta\ AXY = \frac{1}{2}(AX)(AY)\sin\alpha$. Now $\alpha = \arcsin\frac{4}{5}$ so $A_1 = \frac{1}{2}S_1^2\,\frac{4}{5} = \frac{2S_1^2}{5}$. We wish A_1 to be half to total area, i.e., $A_1 = 3$. Setting $\frac{2S_1^2}{5} = 3$ we find $S_1 = \sqrt{\frac{15}{2}}$. Since $\sqrt{\frac{15}{2}} < 3 < 5$, x does indeed lie on $\overline{AC}$ and Y on $\overline{AB}$.

Case 2: Similarly, we can let A_2 = area Δ BXY. Again we can minimize XY by letting $BX = BY = X_2$. $A_2 = \frac{1}{2}S_2^2 \sin\beta$. Since $\beta = \arcsin\frac{3}{5}$, we have $3 = \frac{1}{2}S^2(\frac{3}{5})$ or $S_2^2 = 10$ and $S_2 = \sqrt{10}$. Since $\sqrt{10} < 4 < 5$, x does indeed lie on $\overline{BC}$ and Y on $\overline{AB}$.

4. Case 3: As above, let A_3 = area Δ CXY and let $S_3 = CX = CY$

$$A_3 = \frac{1}{2}(CX)(CY) \sin 90^\circ$$ and substituting:

$$3 = \frac{1}{2} S_3^2 \quad \text{or} \quad S_3 = \sqrt{6}$$

Since $\sqrt{6} < 3 < 4$, x does lie on $\overline{AC}$ and Y on $\overline{BC}$.

Let h be the altitude of each small triangle. Thus

$A = \frac{1}{2}bh$ (where b = xy). Since $h_1 = S_1 \cos\frac{\alpha}{2}$, $h_2 = S_2 \cos\frac{\beta}{2}$,

and $h_3 = S_3 \cos 45^\circ$, we solve:

$$b_1 = \frac{6}{S_1 \cos\frac{\alpha}{2}} = \frac{6}{\sqrt{\frac{15}{2}}\sqrt{\frac{1+\cos\alpha}{2}}} = \frac{6}{\sqrt{\frac{15}{2}}\sqrt{\frac{8}{10}}} = \frac{6}{\sqrt{6}} = \sqrt{6}$$

$$b_2 = \frac{6}{S_2 \cos\frac{\beta}{2}} = \frac{6}{\sqrt{10}\sqrt{\frac{1+\cos\beta}{2}}} = \frac{6}{\sqrt{10}\sqrt{\frac{9}{10}}} = \frac{6}{\sqrt{9}} = 2$$

$$b_3 = \frac{6}{S_3 \cos\frac{\pi}{4}} = \frac{6}{\sqrt{6}\sqrt{\frac{1}{2}}} = \sqrt{12}$$

Clearly $2 < \sqrt{6} < \sqrt{12}$ so the minimum length of $\overline{XY}$ is 2 (Case 2).

5. Find all solutions of the following system of simultaneous equations:

(1) $x + y + z = 7,$ (2) $x^2 + y^2 + z^2 = 31$ (3) $x^3 + y^3 + z^3 = 154.$

$$x^3 + y^3 + z^3 = 154 \Rightarrow x^3 + y^3 = 154 - z^3$$

(A) $(x+y)(x^2-xy+y^2) = 154 - z^3$

(4) From (1) $x + y = 7 - z$

(5) From (2) $x^2 + y^2 = 31 - z^2$

(6) Squaring (4): $x^2 + 2xy + y^2 = 49 + z^2 - 14z$

(7) Subtract (5) and (6): $2xy = 18 + 2z^2 - 14z$

(8) $- xy = -z^2 + 7z - 9$

Add (5) to (8): $x^2 - xy + y^2 = -2z^2 + 7z + 22$

Substitute in (A): $(7 - z)(-2z^2 + 7z + 22) = 154 - z^3$

$$2z^3 - 21z^2 + 27z + 154 = 154 - z^3$$

$$3z^3 - 21z^2 + 27z = 0$$

$$z(z^2 - 7z + 9) = 0$$

So either $z = 0$ or $z = \frac{7 \pm \sqrt{13}}{2}$

If $z = 0$, (1) becomes $x + y = 7$ or $y = 7 - x$

Substituting in (2): $x^2 + (7-x)^2 = 31 \Rightarrow 2x^2 - 14x + 49 = 31$

$x^2 - 7x + 9 = 0 \Rightarrow x = \frac{7 \pm \sqrt{13}}{2}$. If $x = \frac{7 + \sqrt{13}}{2}$,

$y = \frac{7 - \sqrt{13}}{2}$ and vice versa. Checking in

(3): $\left(\frac{7 + \sqrt{13}}{2}\right)^3 + \left(\frac{7 - \sqrt{13}}{2}\right)^3 \stackrel{?}{=} 154$

5. (con'td.)

$$\frac{343 + 147\sqrt{13} + 273 + 13\sqrt{13}}{8} + \frac{343 - 147\sqrt{13} + 273 - 13\sqrt{13}}{8} \overset{?}{=} 154$$

$$\begin{array}{r} 343 \\ 2 \\ \hline 686 \\ 546 \\ \hline 1232 \end{array} \qquad \begin{array}{r} 273 \\ 2 \\ \hline 546 \end{array} \qquad 8\overline{)1232}\ \ 154$$

$$\frac{1232}{8} \overset{?}{=} 154$$

$$154 \overset{!}{=} 154$$

Substituting other values for z merely yields rotations of letters.

So the six solutions are:

$x_1 = 0$ $\qquad y_1 = \frac{7 + \sqrt{13}}{2}$ $\qquad z_1 = \frac{7 - \sqrt{13}}{2}$

$x_2 = 0$ $\qquad y_2 = \frac{7 - \sqrt{13}}{2}$ $\qquad z_2 = \frac{7 + \sqrt{13}}{2}$

$x_3 = \frac{7 + \sqrt{13}}{2}$ $\qquad y_3 = 0$ $\qquad z_3 = \frac{7 - \sqrt{13}}{2}$

$x_4 = \frac{7 + \sqrt{13}}{2}$ $\qquad y_4 = \frac{7 - \sqrt{13}}{2}$ $\qquad z_4 = 0$

$x_5 = \frac{7 - \sqrt{13}}{2}$ $\qquad y_5 = 0$ $\qquad z_5 = \frac{7 + \sqrt{13}}{2}$

$x_6 = \frac{7 - \sqrt{13}}{2}$ $\qquad y_6 = \frac{7 + \sqrt{13}}{2}$ $\qquad z_6 = 0$

Note: One can see immediately that the system has no solution in integers x, y, and z. Since $31 \equiv 7 \bmod 8$ has no solutions. This is based on the fact that $a^2 \equiv 0, 1,$ or $4 \bmod 8$. For all a.