FOURTEENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America, Michigan Colleges and Universities, Professional Organizations, and Industries.

PART I

October 21, 1970

INSTRUCTIONS

(to be read aloud to class by supervisor or proctor)

1. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

2. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

3. Your score on the test will be the number right. You are advised to guess an answer in those cases where you cannot determine the right answer but are able to eliminate some of the alternatives as impossible.

4. Usually a score of 15 or more will allow you to become a finalist and write the second exam. To improve your score, be careful to complete all problems which you can do successfully before working on the other problems. Several questions, distributed throughout the test, can be answered by using only the first two years of high school mathematics.

5. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Do not make calculations on the answer sheet.

6. In each of the questions five different possible responses are proposed. In many cases the fifth alternative is listed "E none of the above." In such cases if you believe none of the first four alternatives to be correct, mark E.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. If you have questions concerning the instructions ask them now.

MICHIGAN MATHEMATICS PRIZE COMPETITION (PART I)

14th ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

1. If $5^{18} + 2$ is divided by 4, the remainder is

   (a) 0 (b) 1 (c) 2 (d) 3 (e) none of these.

2. If the logarithm of $\sqrt{27}$ to the base b is $3/2$, the number b is

   (a) 2 (b) 3 (c) 10 (d) 9 (e) none of these.

3. If $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$, then

   (a) necessarily $a=b$.

   (b) necessarily $a=0$.

   (c) necessarily $a=0$ or $b=0$.

   (d) necessarily $a=0$ and $b=0$.

   (e) none of these.

4. If $|a + b| = |a| + |b|$, then we can conclude that

   (a) a and b are positive.

   (b) a or b is zero.

   (c) a and b are negative.

   (d) a and b are of opposite sign.

   (e) none of these.

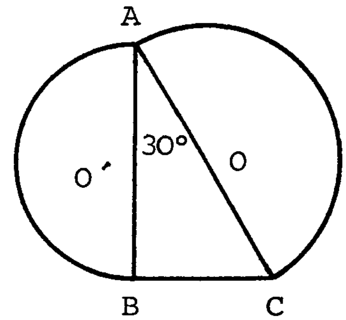
5. A circle is revolved about a diameter and the sphere obtained has numerically the same volume as the circle had area. The circumference of the circle is

   (a) 6π (b) $3\pi/2$ (c) $3/4$ (d) $3/2$ (e) none of these.

6. In the figure $\triangle ABC$ is a right triangle with $m(\angle A) = 30°$. The ratio of the area of semi-circle O to the area of semi-circle O' is

(a) $4\pi/3$ (b) $5/4$ (c) $4/3$

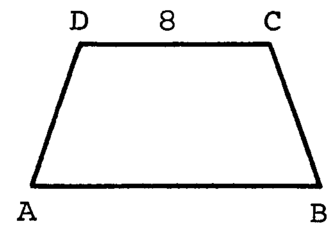
(d) $3/4$ (e) none of these.



7. Given the isosceles trapezoid ABCD. If DC = 8, the height is 6, and the area is 60, then AD =

(a) $2\sqrt{10}$ (b) $2\sqrt{13}$ (c) 5

(d) $3\sqrt{5}$ (e) none of these.



8. Pieces of paper numbered 1 through 10 are mixed in a jar and then two are drawn at random. The probability that the sum of the numbers drawn is odd is

(a) 1/2 (b) 1/4 (c) 3/8 (d) 5/9 (e) none of these.

9. AB is tangent to circle O at A. AB = 8 and BC = 5. $\theta =$

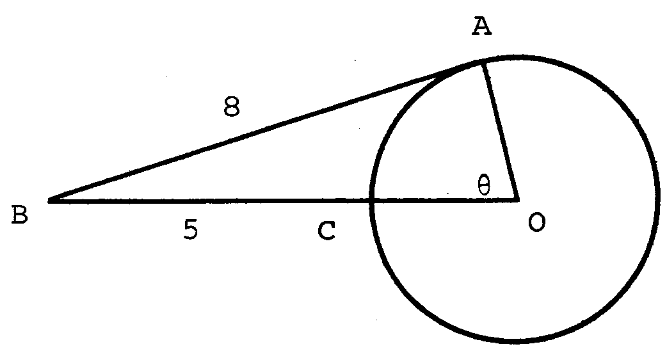
(a) $\cot^{-1} 49/99$

(b) $\tan^{-1} 80/39$

(c) $\sec^{-1} 5/8$

(d) $\cot^{-1} 80/39$

(e) none of these



10. The solution set in the real numbers of the open sentence $\sqrt{7x} = \sqrt{3x^2-6}$ is

(a) $\{-2/3, 3\}$ (b) the empty set (c) $\{3\}$ (d) $\{2/3, 3\}$

(e) none of these.

11. The sum of the squares of the roots of the equation $x^2 + (m-2)x - (m+3) = 0$ is equal to

(a) -5 (b) $2-m$ (c) $m+3$ (d) $(m-1)^2 + 9$

(e) none of these.

12. In the equation $x^2 - \frac{15}{4}x + a^3 = 0$ one of the roots is the square of the other if a is equal to

(a) 2 or 3 (b) 5 or -3 (c) $-2/5$ or $2/3$ (d) $-5/2$ or $3/2$

(e) none of these.

13. For each number a, the line $y = (2a+3)x - a^2$ intersects the curve $y = x^2 + 3x$ in precisely one point. The line intersecting the curve at the point $(2/3, 22/9)$ has the equation

(a) $y = x - 16/9$ (b) $y = (2/3)x - 18/9$

(c) $y = (13/3)x - 4/9$ (d) $y = (3/2)x + 13/9$

(e) $y = (4/9)x - 58/27$.

14. According to regulations a Greyhound bus driver should cover a distance between two stops at an average speed of 48 m.p.h. However, the first half of a certain distance he drove at 60 m.p.h. In order to comply with regulations his average speed for the second half should be

(a) 36 m.p.h. (b) 40 m.p.h. (c) 50 m.p.h.

(d) 45 m.p.h. (e) none of these.

15. If $\cos\alpha = \frac{2ab}{a^2+b^2}$, then $\tan^2\frac{\alpha}{2} =$

(a) $\left(\frac{a-b}{a+b}\right)^2$ (b) $\frac{a^2-b^2}{2ab}$ (c) $\left(\frac{a+b}{a-b}\right)^2$ (d) $\frac{2ab}{a^2-b^2}$

(e) none of these.

16. The sum of $\frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \ldots + \frac{1}{\log_{100} N}$ is

(a) $\frac{\ln 100}{\ln N}$ (b) $\frac{\ln N}{\ln 100!}$ (c) $\log_{100!} N$

(d) $\log_N(2+3+\ldots+100)$ (e) $\log_N 2 + \log_N 3 + \ldots + \log_N 100$.

17. The product $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \ldots \log_{255} 256$ is equal to

(a) 16 (b) 8 (c) $\log_8 256$ (d) $\log_{128} 256$

(e) none of these.

18. If $f(x) = \sqrt{x-1} + \sqrt{x+24 - 10\sqrt{x-1}}$, $x \geq 1$, then the graph of the function is

(a) a half line (b) a part of a parabola

(c) a line segment and a part of a parabola

(d) a line segment and a circular arc (e) none of these.

19. If a point C divides the semi-circular arc AB of radius 10 in such a way that the product of the chords AC and BC is 100, then the central angles are

(a) 90° and 90° (b) 60° and 120° (c) 45° and 135°

(d) 30° and 150° (e) none of these.

20. What is the largest positive integer which divides $n^5 - 5n^3 + 4n$ for every positive integer n?

(a) 20 (b) 30 (c) 40 (d) 60 (e) 120.

21. If (x,y) are the rectangular cartesian coordinates of a point in a plane, then

$$\text{maximum}\ (|x|,\ |y|) = 1$$

is the equation of

(a) a circle (b) a triangle (c) a square (d) a line

(e) none of these.

22. A letter is chosen at random from each of the words "ASSININE" and "ASSASSIN". The probability that the same letter is chosen from both words is

(a) 2/9 (b) 9/20 (c) 1/4 (d) 5/16 (e) 7/32.

23. If p is a prime number greater than 7, then p^6-1 is divisible by

(a) $3 \cdot 4 \cdot 5$ (b) $4 \cdot 5 \cdot 6$ (c) $5 \cdot 6 \cdot 7$ (d) $7 \cdot 8 \cdot 9$

(e) $9 \cdot 10 \cdot 11$

24. How many times must a single die be cast to insure at least an even chance that a one ("ace") turns up at least once?

(a) 1 (b) 2 (c) 3 (d) 4 (e) none of these.

25. The repeating decimal .429642964296... represents the fraction

(a) $\frac{537}{1250}$ (b) $\frac{125}{537}$ (c) $\frac{1432}{3333}$ (d) $\frac{333}{1432}$ (e) none of these.

26. January 19, 1944 fell on a

(a) Monday (b) Wednesday (c) Friday (d) Saturday

(e) none of these.

27. If r is a real root of the equation $3x^4 - 61x^3 + 127x^2 + 220x - 520 = 0$, then

(a) $4 \le r \le 9$ (b) $-9 \le r \le 4$ (c) $9 \le r \le 72$

(d) $-4 \le r \le 18$ (e) $-18 \le r \le 9$.

28. Martians, as is well known, have 6 fingers on each of two hands. If a Martian told you that there were 549 sheep in a pasture, when you counted them you would expect to find

(a) 549 (b) 213 (c) 777 (d) 2313 (e) 399

29. How many subsets of a twelve element set are there with an odd number of elements?

(a) 2^{12} (b) 1024 (c) 2048 (d) 12! (e) none of these.

30. If $f(x) = \frac{x-1}{x+1}$, what is the value of $f(f(f(2)))$?

(a) 3 (b) -3 (c) -1/2 (d) 1/3 (e) none of these.

31. If $x^{15} + x^2 + 3$ is divided by $x-1$, the remainder is

(a) 0 (b) 1 (c) 3 (d) 5 (e) none of these.

32. The numeral in the units place when 7^{13} is expressed in base 6 notation is

(a) 5 (b) 4 (c) 3 (d) 1 (e) none of these.

33. The points in the x-y plane whose coordinates satisfy both of the conditions $x+y = \sqrt{2}$ and $x^2+y^2 < 1$ constitutes a set consisting of

(a) exactly two points (b) exactly one point

(c) a circle (d) a line segment (e) none of these.

34. For what real x is $\left|\frac{x+2}{x-3}\right| > 1$?

(a) all $x < 1/2$ (b) all $x \neq 3$ (c) all $x > 1/2$, $x \neq 3$

(d) $1/2 < x < 5\ 1/2$, $x \neq 3$ (e) none of these.

35. If $ac \neq 0$, then an equation whose roots are reciprocals of the roots of $ax^2 + bx + c = 0$ is

(a) $\frac{1}{a}x^2 + \frac{1}{b}x + \frac{1}{c} = 0$ (b) $cx^2 + bx + a = 0$

(c) $ax^2 + cx + b = 0$ (d) $bx^2 + ax + c = 0$

(e) none of these.

36. If the constant term of $(2x - \frac{1}{x})^n$ is -160, then n equals,

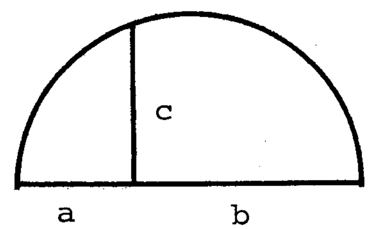
(a) 2 (b) 4 (c) 6 (d) 8 (e) 10

37. In a certain number system one counts as follows:

1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33, 111, ... What is the symbol for the 54^{th} number?

(a) 2000 (b) 1300 (c) 1230 (d) 1223 (e) none of these.

38. If a+b is the diameter of a semi-circle and c makes a right angle with a and b, then c equals



(a) $\sqrt{ab}$ (b) $\frac{a+b}{2}$ (c) $\sqrt{a^2+b^2}$

(d) $\sqrt{a+b}$ (e) none of these.

39. A can circle a track in $2\frac{1}{2}$ minutes, B in $2\frac{3}{4}$ minutes. How many minutes must A drive in order to lap B?

(a) $25\frac{1}{2}$ (b) $27\frac{1}{2}$ (c) 26 (d) $24\frac{3}{4}$ (e) none of these.

40. If $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2}{\frac{1}{a}+\frac{1}{b}}$, a and b positive, which of the following is always true?

(a) $A \geq G \geq H$ (b) $G \geq H \geq A$ (c) $A \geq H \geq G$

(d) $H \geq G \geq A$ (e) $H \geq A \geq G$

The following Michigan companies and professional organizations have made contributions to the scholarship fund for this competition.

Aeroquip Corporation, Jackson
Burroughs Corporation, Detroit
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The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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