THIRTEENTH ANNUAL
MICHIGAN MATHEMATICS PRIZE COMPETITION

Part II

December 10, 1969

Solutions

The solutions credited to students are verbatim reproductions except for minor editing.

1. Two trains, A and B, travel between cities P and Q. On one occasion A started from P and B from Q at the same time and when they met A had travelled 120 miles more than B. It took A four (4) hours to complete the trip to Q and B nine (9) hours to reach P.

Assuming each train travels at a constant speed, what is the distance from P to Q?

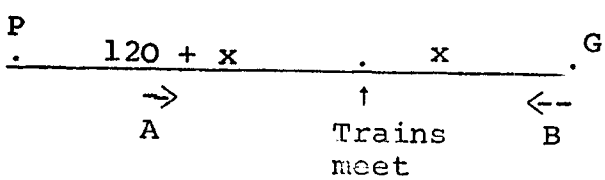
Problem 1

Comment. The problem is ambiguously worded. The third sentence should read: It took A four (4) hours after meeting B to complete the trip to G and it took B nine (9) hours after meeting A to reach P.

The solution given below is based on this interpretation. Any reasonable interpretation was accepted and full credit given if solved correctly.

Solution 1 by Kendall Barker, Grosse Pt. South.

The ratio between the rate of A and the rate of B is $\frac{x + 120}{x}$.



A's rate is $\frac{x}{4}$ miles per hour.

B's rate is $\frac{120 + x}{9}$

Hence $\frac{x + 120}{x} = \frac{x}{4} / \frac{120 + x}{9} = \frac{9x}{4(120+x)}$

$\Rightarrow 9x^2 = 4(120 + x)^2 \Rightarrow 3x = \pm\, 2(120 + x)$.

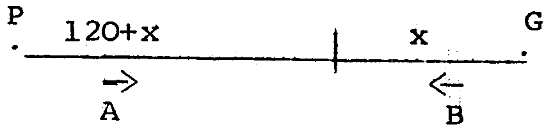
Since x must be positive $3x = 240 + 2x \Rightarrow x = 240$.

Thus the distance from P to Q is $2x + 120$ or 600 miles.

Problem 1

<u>Solution 2</u> by James Ellis, Hillsdale H.S.

Diagram:



I have some question about this problem. If "complete the trip" means the whole trip from P to Q rather than the completion of the trip after the trains have met (as I figured below) then I got 312 miles from P to Q.

Let r_A = rate of train A; r_B = rate of train B both in m.p.h.

Since train A left P at the same time B left Q, when the trains meet each has travelled the same length of time. Let t equal this time in hours. Because train A takes 4 hours to complete its run (that is to cover the distance B travels to the meeting point) $4\ r_A = t\ r_B$

Similarly $9\ r_B = t\ r_A$

Multiplying equation (1) by t gives $4\ tr_A = t^2 r_B$

Multiplying equation 2 by 4 gives $36\ r_B = 4\ tr_A$

Since $r_B > 0$, solving simultaneously gives $t^2 = 36$ or $t = 6$.

Thus from equation (1) we get $4r_A = 6r_B$ or $r_A = \frac{3}{2} r_B$.

This implies that if the trains run for the same length of time, A will cover $\frac{3}{2}$ the distance covered by B.

Hence $\frac{3}{2}x = 120 + x \Rightarrow x = 240$ and the distance from P to Q is 600 miles.

2. If a and b are integers, b odd, prove that

$$x^2 + 2ax + 2b = 0 \text{ has no rational roots.}$$

Solution 1. Toshihiko Fukuyama Kimball H.S.

From the quadratic formula we have $x = -a \pm \sqrt{a^2 - 2b}$

If x is rational $a^2 - 2b = r^2$, r a rational number, in fact an integer.

$$a^2 - r^2 = 2b \Rightarrow (a+r)(a-r) = 2b.$$

Since b is an odd integer both $a + r$ and $a - r$ cannot both be even, hence one must be even and one odd.

But the sum of an even and an odd number is odd while the sum of $a + r$ and $a - r$ is $2a$ an even integer. This contradiction means that the original equation has no rational root.

2nd Solution

If the fraction $\frac{p}{q}$ in lowest terms is a root of the equation then $\frac{p^2}{q^2} + 2a\frac{p}{q} + 2b = 0$ and $p^2 = -2apq - 2bq^2 = -2q(ap + bq)$. Thus, p^2 is even and thus must have a factor 4. Furthermore, since $\frac{p}{q}$ is in lowest terms, q is odd. Also ap is even and bq odd so $ap + bq$ is odd. Hence $-2q(ap + bq)$ cannot be divisible by 4.

This is a contradiction and shows that the equation cannot have a rational root.

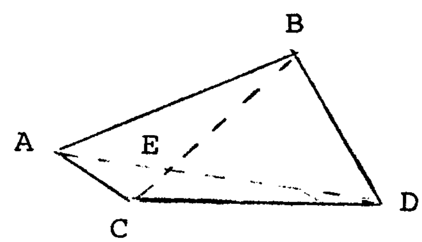
3. A diameter segment of a set of points in a plane is a segment joining two points of the set which is at least as long as any other segment joining two points of the set.

Prove that any two diameter segments of a set of points in the plane must have a point in common.

<u>Solution</u> Paul Metzner, Ann Arbor - Pioneer

If the diameter segments AB and CD do not intersect there are two cases to consider.

<u>Case 1</u>. If AD intersects BC at E.



Case 1

$\overline{AE} + \overline{EB} > \overline{AB}$ $\quad \overline{CE} + \overline{ED} > \overline{CD}$

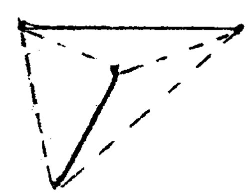
$\therefore \overline{AE} + \overline{EB} + \overline{CE} + \overline{ED} > \overline{AB} + \overline{CD}$

Since $\overline{AE} + \overline{ED} = \overline{AD}$ and $\overline{CE} + \overline{EB} = \overline{BC}$

$\overline{AD} + \overline{BC} > \overline{AB} + \overline{CD} = 2\ \overline{AB}$

$\Rightarrow$ either $\overline{AD}$ or $\overline{BC}$ greater than $\overline{AB}$ and AB is not a diameter segment.

<u>Case 2.</u> AD doesn't intersect BC;
then CD lies within Δ ABD or Δ ABC <u>or</u> AB lies within Δ CDA or Δ CDB.



Case 2

All four possibilities are essentially the same, so I will simply consider the one case where CD is within Δ ABD.

One of the angles ACD or BCD is obtuse. Thus, with $\overline{AD}$ or $\overline{DB}$ is greater than $\overline{CD}$. This means CD is not a diameter segment.

Thus the assumption that the diameter segments AB and CD do not intersect is untenable.

4. Find all positive integers n for which

$$\frac{n(n^2 + n + 1)(n^2 + 2n + 2)}{2n + 1}$$

is an integer. Prove that the set you exhibit is complete.

Solution Mordecai Abramowitz Stevenson H.S., Livonia

Suppose

$$I = \frac{n(n^2 + n + 1)(n^2 + 2n + 2)}{2n + 1} = \frac{n^5 + 3n^4 + 5n^3 + 4n^2 + 2n}{2n + 1}$$

By long division this equals:

$$\frac{1}{2} n^4 + \frac{5}{4} n^3 + \frac{15}{3} n^2 + \frac{17}{16} n + \frac{15}{32} + \frac{15}{32} \,/\, 2n + 1$$

So

$$32\,I = 16\,n^4 + 40\,n^2 + 60\,n^2 + 34n + 15 + \frac{15}{2n+1}$$

Hence $\frac{15}{2n + 1}$ is an integer.

Thus $2n + 1$ must be either 1, 3, 5, or 15 .

$2n + 1 = 1 \Rightarrow n = 0$ n is not a positive integer.

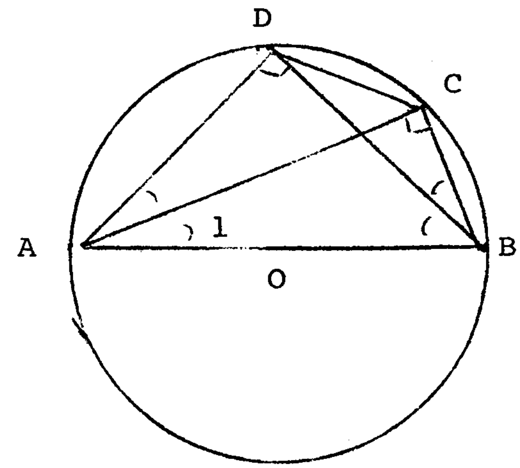
$2n + 1 = 3 \Rightarrow n = 1$ $\frac{1\,(3)(5)}{3} = 5$ OK

$2n + 1 = 5 \Rightarrow n = 2$ $\frac{2(7)(10)}{5} = 28$ OK

$2n + 1 = 15 \Rightarrow n = 7$ $\frac{7(57)(65)}{15} = 7(19)(13)$ OK

Hence $n = \{1, 2, 7\}$.

5. A, B, C, D are four points on a semi circle with diameter $\overline{AB} = 1$. If the distances $\overline{AC}$, $\overline{BC}$, $\overline{AD}$, $\overline{BD}$ are all rational numbers, prove that $\overline{CD}$ is also rational.



<u>Solution 1</u> Kevin Butler, Cranbrook H.S.

Since ABD and ACB are right triangles:

$$\sin x = \frac{AD}{1} = AD \qquad \cos x = BD$$
$$\sin y = AC \qquad \cos y = BC$$

$$\sin y = \sin(y-x) = \sin y \cos x - \cos y \sin x = AC \cdot BD - BC \cdot AD$$

By the sine law applied to Δ DCB.

$\frac{CD}{\sin y} = 2R$, R the radius of the circle circumscribing triangle DCB .

$$\frac{CD}{AC \cdot BD - BC \cdot AD} = 1 \Rightarrow CD = AC \cdot BD - BC \cdot AD$$

Thus CD is rational.

<u>Solution 2</u> Mark Marchine, Seaholm H.S.

If AC and BC are both rational, as is AB, then we know that all the trig functions of the angles in right Δ ABC are rational. Similarly all trig functions of angles is Δ ADB are rational also. Recalling that $\sin(a-b) = \sin a \cos b - \cos a \sin b$, then $\sin \angle DAC$ is rational.

Now since $\frac{a}{\sin a} = \frac{b}{\sin b} = \frac{c}{\sin c}$ = diameter circumscribing circle, $\frac{DC}{\sin \angle DAC} = AB \Rightarrow DC$ is rational.

<u>Comment</u> A well known theorem of Ptolemy asserts that if ABCD are the consecutive vertices of an inscribed convex quadrilateral that $AB \cdot CD + BC \cdot AD = AC \cdot BD$. The solution of this problem follows easily from this fact.