# THIRTEENTH ANNUAL

# MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America, Michigan Colleges and Universities, Professional Organizations, and Industries.

## PART I

October 22, 1969

### INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

2. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

3. Your score on the test will be the number right. You are advised to guess an answer in those cases where you cannot determine the right answer but are able to eliminate some of the alternatives as impossible.

4. Usually a score of 15 or more will allow you to become a finalist and write the second exam. To imporve your score, be careful to complete all problems which you can do successfully before working on the other problems. Several questions, distributed throughout the test, can be answered by using only the first two years of high school mathematics.

5. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Do not make calculations on the answer sheet.

6. In each of the questions five different possible responses are proposed. In many cases the fifth alternative is listed "E none of the above". In such cases if you believe none of the first four alternatives to be correct, mark E.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. If you have questions concerning the instructions ask them now.

13th ANNUAL MICHIGAN MATHEMATICS

PRIZE COMPETITION

1. The number of terms in the expansion of $[(a+3b)^2 (a-3b)^2]^2$ when simplified is

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

2. A formula expressing the relationship between x and y has corresponding values shown in the accompanying table.

| x | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| y | 100 | 90 | 70 | 40 | 0 |

A possibility for this formula is

(a) $y = 100 - 10x$ (b) $y = 100 - 5x^2$

(c) $y = 100 - 5x - 5x^2$ (d) $y = 20 - x - x^2$

(e) none or more than one

3. A pair of factors of $x^4 + 9$ is

(a) $(x^2 + 3)\ (x^2 + 3)$ (b) $(x^2 + 3)\ (x^2 - 3)$ (c) $(x^2 + 9)x^2$

(d) $(x^2 - 3x + 3)\ (x^2 + 3x + 3)$ (e) none of these

4. When simplified $(a^{-1} + b^{-1})^{-1}$ is equal to

(a) $a + b$ (b) $\frac{ab}{a+b}$ (c) $ab$ (d) $\frac{1}{ab}$ (e) none of these

5. A man buys a house for $10,000 and rents it. He puts $12\frac{1}{2}$ per cent of each months rent aside for repairs and upkeep, pays $325 a year taxes and realized $5\frac{1}{2}$ per cent profit on his investment. The monthly rent (to the nearest penny) is

(a) $64.82 (b) $83.33 (c) $72.08 (d) $45.83 (e) $177.08

6. Given $\triangle ABC$ with D, E, F the midpoints of its sides while G, H and I are the midpoints of the sides of $\triangle DEF$. If the area of $\triangle ABC$ is 400 square inches then the area of $\triangle GHI$ is

(a) 100 square inches (b) 25 square inches (c) 50 square inches

(d) 1600 square inches (e) none of these

7. A car travels 120 miles from A to B at 30 m.p.h. but returns the same distance at 40 m.p.h. The average speed for the round trip is closest to

(a) 33 mph (b) 34 mph (c) 35 mph (d) 36 mph (e) 37 mph

8. After rationalizing the numerator of the fraction $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{3}}$ the demoninator of the resulting fraction in simplified form is

(a) $\sqrt{3}\,(\sqrt{3} + \sqrt{2})$ (b) $\sqrt{3}(\sqrt{3} - \sqrt{2})$ (c) $3 - \sqrt{3}\sqrt{2}$ (d) $3 + \sqrt{6}$

(e) none of these

9. At 2:15 o'clock the hour and the minute hands of a clock form an angle of

(a) $30^\circ$ (b) $5^\circ$ (c) $22\frac{1}{2}^\circ$ (d) $7\frac{1}{2}^\circ$ (e) $28^\circ$

10. If the length of a diagonal of a square is a + b then the area of the square is

(a) $(a + b)^2$ (b) $\frac{1}{2}(a + b)^2$ (c) $a^2 + b^2$ (d) $\frac{1}{2}(a^2 + b^2)$

(e) none of these

11. The difference between the roots of $x^2 - 7x - 9 = 0$ is

(a) 7 (b) $\frac{7}{2}$ (c) 9 (d) $2\sqrt{85}$ (e) $\sqrt{85}$

12. When $x^{13} + 1$ is divided by $x - 1$ the remainder is

(a) 1 (b) -1 (c) 0 (d) 2 (e) none of these

13. If $y = f(x) = \frac{x+2}{x-2}$ then it is correct to say

(a) $x = \frac{y+2}{y-1}$ (b) $f(0) = -2$ (c) $f(1) = 0$ (d) $f(-2) = 0$

(e) $f(y) = x$

14. Given 12 points in the plane, no three collinear, then the number of lines they determine is

(a) 24 (b) 54 (c) 120 (d) 66 (e) none of these

15. The sum of the squares of the roots of $x^2 - 3x + 1 = 0$ is

(a) a positive integer (b) a positive fraction greater than 1

(c) a positive fraction less than 1. (d) an irrational number

(e) an imaginary number

16. The number of subsets containing four or fewer elements from a set of twelve elements is

(a) 1365 (b) 794 (c) 495 (d) 496 (e) none of these

17. If the circumference of a circle is $\pi^2$ then the length of the side of a square inscribed in the circle is

(a) 2 (b) $\frac{\sqrt{2}\,\pi}{2}$ (c) $\frac{\sqrt{2}}{2\,\pi}$ (d) $\frac{3\pi}{2}$ (e) none of these

18. $S_1$ and $S_2$ are externally tangent spheres and P is a plane. Then

(a) $P \cap (S_1 \cup S_2)$ is never a point

(b) $P \cap (S_1 \cup S_2)$ is never two tangent circles

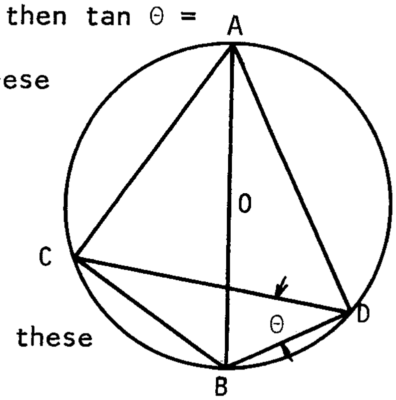
(c) $P \cap (S_1 \cup S_2)$ is never two points

(d) $P \cap (S_1 \cup S_2)$ is never two non-intersecting circles

(e) none of these

19. If in the circle pictured, the radius $\overline{OA} = 5$ while $\overline{CB} = 6$ then $\tan \Theta =$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{4}{3}$ (d) 2 (e) none of these




20. The probability of rolling an 11 on a single roll of two standard dice is

(a) $\frac{2}{9}$ (b) $\frac{1}{12}$ (c) $\frac{1}{36}$ (d) $\frac{1}{18}$ (e) none of these

21. The number of real roots $2x^3 - 6x^2 - 1 = 0$ is exactly

(a) 4 (b) 3 (c) 2 (d) 1 (e) 0

22. By adding the same constant to each of the numbers 20, 50, 100 a geometric progression results. The common ratio is

(a) $\frac{5}{3}$ (b) $\frac{4}{3}$ (c) $\frac{3}{2}$ (d) $\frac{1}{2}$ (e) $\frac{1}{3}$

23. Cos [2 arc sec 2] =

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 0 (d) $\frac{1}{\sqrt{2}}$ (e) none of these

24. In the set of real numbers the complete solution set for

$$\begin{vmatrix} x & 0 & 1 \\ x^2 & 3 & 4 \\ 0 & 3 & 5 \end{vmatrix} = 0, \text{ is}$$

(a) {0} (b) {0,1} (c) {0,-1}

(d) {-1} (e) none of these

25. The radius of a right circular cylinder is r and the height is h. If the radius is increased by $\delta$, then in order for the volume not to change the height must be decreased by

(a) $\delta$ (b) $\frac{(2r + \delta)h}{(r + \delta)^2}$ (c) $\pi\delta^2$ (d) $\frac{(2r + \delta)\ \delta h}{(r + \delta)^2}$

(e) none of these

26. A fair coin is tossed 5 successive times. What is the probability that at least one head is obtained?

(a) 8/10 (b) 9/10 (c) 1 (d) 31/32 (e) none of these

27. The number of distinct points common to the graphs of $x^2 + y^2 = 16$ and $y^2 = 9$ is

(a) 0 (b) 1 (c) 2 (d) 4 (e) infinitely many

28. If the diagonal of a cube is 1, the surface area of a cube is

(a) 1 (b) 2 (c) $\sqrt{\frac{3}{2}}$ (d) $\sqrt{3}$ (e) none of these

29. The pair of equations $3^{x+y} = 81$ and $81^{x-y} = 3$ has

(a) no common solution (b) the solution $x = 2$, $y = 2$

(c) the solution $x = 2\frac{1}{2}$, $y = 1\frac{1}{2}$ (d) a common solution in positive and negative integers

(e) none of these

30. The infinite decimal 2.525252... can be written as a fraction. When reduced to lowest terms the sum of the numerator and denominator of this fraction is

(a) 7 (b) 29 (c) 141 (d) 349 (e) none of these

31. If $2x + y > 12$ and $x + 2y < 9$ then

(a) $x < 5$ and $y < 2$ (b) $x > 5$ and $y > 2$ (c) $x > 2$ and $y > 5$

(d) $x < 2$ and $y < 5$ (e) none of these

32. If $x(a+b) + y(a-b) = 2$ and $ax + by = \frac{a^2+b^2}{a^2-b^2}$ then $\frac{1}{x} + \frac{1}{y} =$

(a) 0 (b) 2a (c) 2b (d) $a^2-b^2$ (e) none of these

33. The differences between the hypotenuse and the other two sides of a right triangle are respectively 8 feet and 4 feet. The perimeter of the triangle is

(a) 48 (b) 36 (c) 50 (d) 37 (e) none of these

34. If $n$ is a whole number then the largest number that $n(n+1)(2n+1)$ is divisible by for all $n$ is

(a) 2 (b) 6 (c) 10 (d) 3 (e) none of these

35. The $x$ and $y$ coordinates of each end point of a line segment of length $\sqrt{2}$ satisfy: maximum $\{|x|, |y|\} = 1$. The minimum distance from the origin to any such line segment is closest to

(a) .7 (b) .8 (c) .9 (d) .6 (e) 1.0

36. The polynomial $x^4 + 2x^3 - x + \frac{1}{4}$ has a polynomial square root. The absolute value of the sum of its coefficients is equal to

(a) $1\frac{1}{2}$ (b) $2\frac{1}{4}$ (c) $2\frac{7}{8}$ (d) 0 (e) none of these

37. If $f$ is a function given by $f(x) = 2x + 3$ and if $f^2(x) = f(f(x))$ and in general if $f^n(x) = f(f^{n-1}(x))$ then $f^n(x)$ is

(a) $4x + 9$ (b) $(2x + 3)^n + 3$ (c) $2^n x + 3(2^n - 1)$

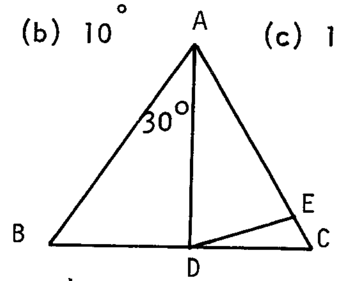
(d) $2^n x + 3^{n-1}$ (e) none of these

38. If the product $2^{1/4} \cdot 4^{1/8} \cdot 8^{1/16} \cdot 16^{1/32} \ldots$ be continued till there are one million factors then the value of the product will be extremely close to

(a) $\sqrt{2}$ (b) 2 (c) 4 (d) 32 (e) none of these

39. If in the figure $\overline{AB} = \overline{AC}$, $\angle BAD = 30°$ and $\overline{AE} = \overline{AD}$, the $\angle EDC$ equals

(a) $7\frac{1}{2}°$ (b) $10°$ (c) $12\frac{1}{2}°$ (d) $15°$ (e) $20°$




40. If $S = \{(x,y):$ x and y are non-zero integers$\}$ is a subset of the X-Y plane, then

(a) every straight line through (0,0) has a point in common with S.

(b) The straight line through (0,0) and (5,8) meets S in a finite number of points.

(c) There is a straight line intersecting S in precisely two points.

(d) There is a straight line through (0,0) which does not intersect S.

(e) None or more than one of the above statements are valid.

The following Michigan companies and professional organizations have made contributions to the scholarship fund for this competition.

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