E. A. Nordhaus

IDENTIFICATION
NUMBER

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TWELFTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

Grade
Sat Jan 18th

sponsored by

The Michigan Section of the Mathematical Association of America, Michigan Colleges and Universities, Professional Organizations, and Industries.

PART II

January 8, 1969

INSTRUCTIONS
(To be read aloud to class by supervisor or proctor)

1. Record, in the upper lefthand corner of this page, the identification number from your questionnaire form. This is the only way to identify this test booklet with your name. Please do not write your name on the booklet.

2. Part II consists of problems and proofs. You will be allowed 100 minutes for the five questions.

3. You are not expected to solve all the questions completely. Look over all problems and work first on those which interest you the most.

4. Each problem is on a different page. You should show most of your work on that page. If it is necessary to use additional paper for your answer, please indicate clearly your identification number and problem number in the upper lefthand corner of each sheet.

5. If you are unable to solve a particular problem, partial credit might be given for indicating a possible procedure or an example to illustrate the ideas involved.

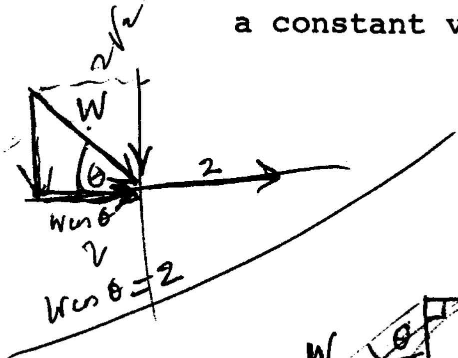
6. You are advised to consider specializing or generalizing any problem where it seems appropriate. Sometimes an examination of special cases may generate ideas of how to attack the problem. On the other hand, a carefully stated generalization may justify additional credit provided you give an explanation of why the generalization might be true.

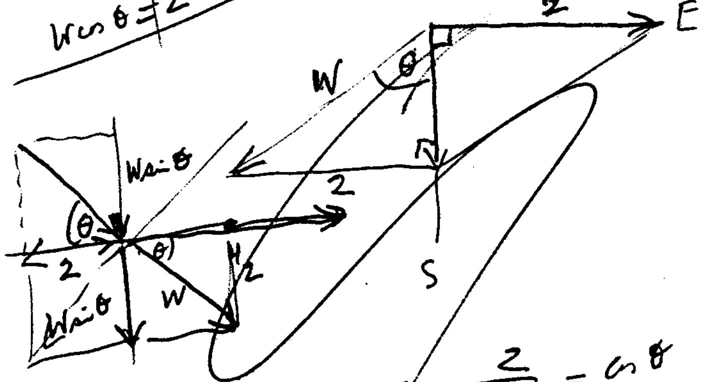
7. Your supervisor is not permitted to violate the rules by answering any questions. When the supervisor announces that the 100 minutes are up, please cease work immediately and insert all significant extra paper, including the questionnaire form, into the booklet. It is not necessary to return scratch paper on which routine numerical calculations were made.

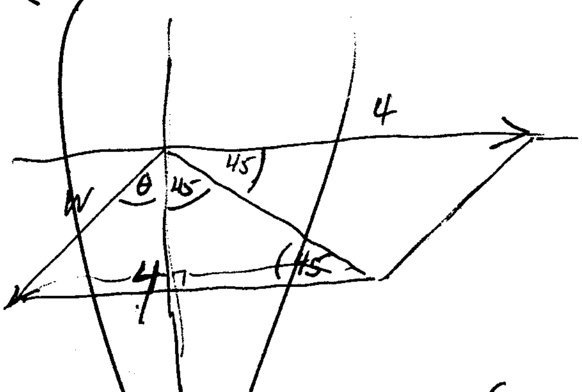
Score	____	____	____	____	____	____
	1	2	3	4	5	Total

I. A man is walking due east at 2 m.p.h. and to him the wind appears to be blowing from the north. On doubling his speed to 4 m.p.h. and still walking due east, the wind appears to be blowing from the northeast. What is the speed of the wind (assumed to have a constant velocity)?

(Take diff of real vectors)







$\theta = 45$

$W\sin\theta = 2$

$\sin\theta = \frac{2}{W} = \cos\theta$

$\frac{W}{\sin 45} = \frac{4}{\sin(\theta+45)}$

$\left(\sin 45 = \cos 45 = \right.$

$W = \frac{2}{\sin\theta} = \frac{4\sin 45}{\sin\theta\cos 45 + \cos\theta\sin 45} = \frac{4}{\sin\theta + \cos\theta}$

wind coming from NW at $2\sqrt{2}$ mph

$2\sin\theta + 2\cos\theta = 4\sin\theta$

$2\cos\theta = 2\sin\theta$

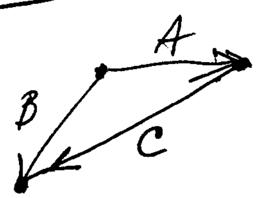
$\tan\theta = 1$ $\qquad \underline{\theta = 45^\circ}$

$\therefore \underline{W = 2\sqrt{2}}$ mph $\doteq 2.828$ mph

See last page

$4\sqrt{2}$ 4
2 W
2

$W^2 = 8$
$W = 2\sqrt{2}$

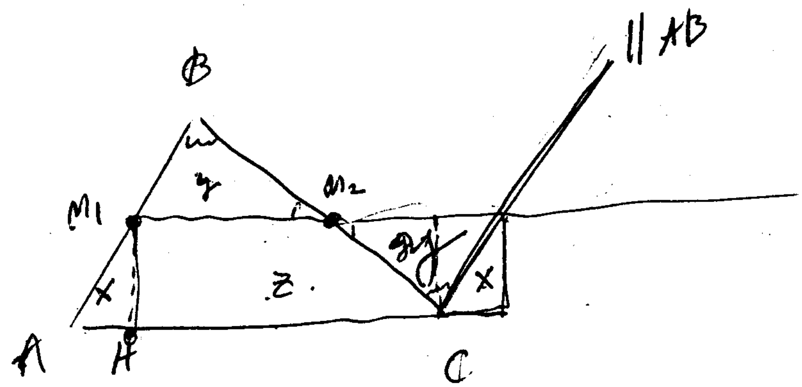


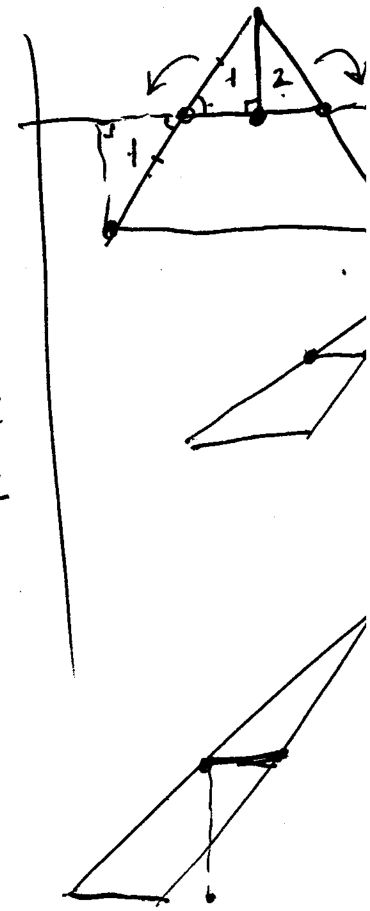
$A + C = B$

$\underline{C = B - A}$

end of A to end of B $= \bar{B} - \bar{A}$

II. Prove that any triangle can be cut into three pieces which can be rearranged to form a rectangle of the same area.





Use <u>longest side</u> as base to avoid M_1H falling outside AC

Part II

18th. Sat.

Seidel — Mar. 29th 8?
gr thy + finite proj planes
(between teams)
error correcting codes?

Each problem is worth 10 points? (possibly extra points for ~~[illegible]~~ good generalizations)

Drawing a picture (with no explanation) — 0

a good start —

~~[illegible]~~

correct method, some slight errors —

<u>Correct answer</u> but incorrect reasoning — 0

9 10
1, 3, 8 24, 15

III. An increasing sequence of integers starting with 1 has the property that if n is any member of the sequence, then so also are 3n and n + 7. Also, all the members of the sequence are ~~solely~~ generated ~~from the~~ first member 1; thus the sequence starts with 1,3,8,9,10,... and The numbers 2,4,5,6,7,... , are not members of this sequence. Determine all the other positive integers which are not members of the sequence.

n	$3n$	$n+7$
1	3	8
3	9	10
8	24	15
9	27	16
10	30	17
24	72	31
15	45	22

$1 \in S \to 7k+1 \in S$

$3 \in S \to 7k+3 \in S$

$9 \in S \to 7k+2 \in S$

$27 \in S \to 7k+6 \in S$

$11 \cdot 7+4 = 77+4 = 81 \in S \to 7k+4 \in S$

$34 \cdot 7+5 = 238+5 = 243 \in S \to 7k+5 \in S$

Clearly $7k \notin S$ $(k \geq 1)$

34
238
243

11, 12, 13, 14, 18, 19, 20, 21 and $\begin{cases} 25+7k \\ 26+7k \\ 28+7k \end{cases}$ $k = 0, 1, 2, \dots$

11, 12, 13, 14, 18, 19, 20 and $\begin{cases} 7k \\ 7k+4 \\ 7k+5 \end{cases}$ $k \geq 3$

For k suff large, only 7k not in S

Induction:— If $m \in S$ then $m-7$ or $\frac{m}{3} \in S$

IV. Three prime numbers, each greater than 3, are in arithmetic progression. Show that their common difference is a multiple of 6.

$6n+1, \quad \underset{\text{prime}}{6n+1+d}, \quad \underset{\text{prime}}{6n+1+2d}$

$\therefore \quad 1+d \equiv 1 \text{ or } 5 \pmod 6$ and $1+2d \equiv 1 \text{ or } 5 \pmod 6$

$d \equiv 0 \text{ or } 4 \pmod 6$ $\qquad 2d \equiv 0 \text{ or } 4 \pmod 6$

$\therefore \; 2d \equiv 0 \text{ or } 2 \bmod 6$

$\therefore \; d \equiv 0 \pmod 6$

$6k-1, 6l+1, 6n-1$

$\therefore \; 12l+2 = 6k+6n-2$

$\cancel{6(2l-k-n) = -4}$

$3(k+n-2l) = 2$

$*$

$4 = 6(k+n-2l)$

$2 = 3(k+n-2l)$

$6n+5, \; 6n+5+d, \; 6n+5+2d$

$5+d \equiv 1 \text{ or } 5 \pmod 6$ and $5+2d \equiv 1 \text{ or } 5 \pmod 6$

$\therefore \; d \equiv 2 \text{ or } 0 \bmod 6$ $\qquad 2d \equiv 2 \text{ or } 0 \pmod 6$

Then $2d \equiv 4 \text{ or } 0 \bmod 6$

$\therefore \; d \equiv 0 \pmod 6$

(3, 5, 7 excluded since not all primes are > 3)

$d = 2$

$p, q, r \qquad p < q < r$

$d = q - p = r - q$

$2q = p + r$

If d not div by 2, then d odd, $p+d = q$ is even

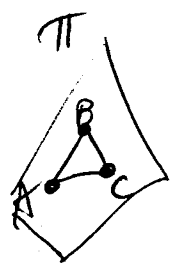
If d not div by 3, then --

$a \; b \; c$

$b - a = c - b$

$b = \frac{a+c}{2}$

V. Prove that if S is a set of at least 7 distinct points, no four in a plane, the volumes of all the tetrahedra with vertices in S are not all equal.



{ plane π thru S^{ABC} for[med] by 3 pts, ∃ other [4] pts on ~~same~~ side of [π]

Four tetrahedra with base AB[C] have = alt. if vols are [=]

But then pts 1, 2, 3, 4 are equid from π & on ~~same~~ side, so lie in a plane ∥ π, a contra.

∴ not all tetrahedra formed can have equal volumes

$$Vol = \frac{1}{3} B h$$

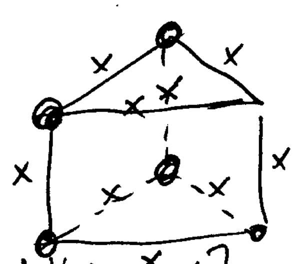
exten pts for possible generalizations?

For 6 pts — can have all tetrahedra of equal vol ?

NO

$$\frac{1}{6}\begin{vmatrix} 1 & 1 & x_1 & y_1 & z_1 \\ 1 & 1 & & & \\ 1 & 1 & & & \\ 1 & 1 & & & \\ 1 & 1 & x_5 & y_5 & z_5 \end{vmatrix} = 0 = \frac{1}{6}\left[\pm V_1 \pm V_2 \pm V_3 \pm V_4 \pm V_5\right]$$

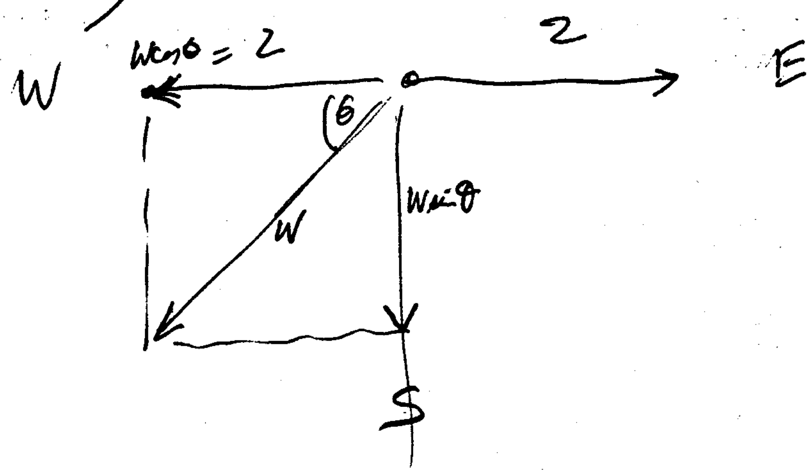
exp. by 1st col

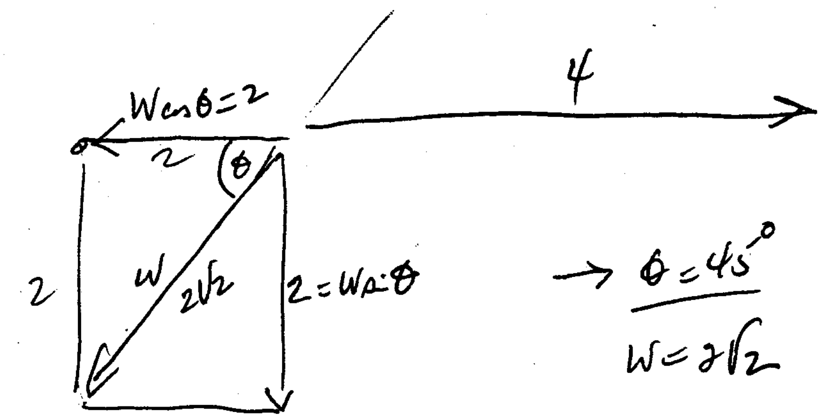


If all =,

$$\frac{1}{6}\left[\pm V \pm V \pm V \pm V \pm V\right] \neq 0$$

I)





$\rightarrow$ $\theta = 45^\circ$

$W = 2\sqrt{2}$

The following Michigan companies and professional organizations have made contributions to the scholarship fund for this competition.

Aeroquip Corporation, Jackson
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Chrysler Corporation, Detroit
Consumers Power Company, Jackson
Electro-Voice, Incorporated, Buchanan
Kuhlman Electric Company, Birmingham
Michigan Council of Teachers of Mathematics
Packaging Corporation of America, Filer City
Walbro Corporation, Cass City

The Annual Awards Banquet is sponsored by Michigan Bell Telephone Company, Detroit.

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