TWELFTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America, Michigan Colleges and Universities, Professional Organizations, and Industries.

PART I

November 20, 1968

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

2. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

3. Your score on the test will be the number right. You are advised to guess an answer in those cases where you cannot determine the right answer but are able to eliminate some of the alternatives as impossible.

4. Usually a score of 15 or more will allow you to become a finalist and write the second exam. To improve your score, be careful to complete all problems which you can do successfully before working on the other problems. Several questions, distributed throughout the test, can be answered by using only the first two years of high school mathematics.

5. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Do not make calculations on the answer sheet.

6. In each of the questions five different possible responses are proposed. In many cases the fifth alternative is listed "E none of the above". In such cases if you believe none of the first four alternatives to be correct, mark E.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. If you have questions concerning the instructions ask them now.

1. Simplify $$\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}}}$$

(a) $\frac{70}{169}$ (b) $\frac{29}{70}$ (c) $\frac{3}{7}$ (d) $\frac{31}{2^6}$

(e) none of the above

2. If A, B, and C are all subsets of S such that $A \supset B \supset C$, then

(a) $A \cap (B \cup C) = C$ (b) $A \cup (B \cap C) = B$

(c) $(A \cap B) \cup C = A$ (d) $A \cap (B \cup C) = B \cup (A \cap C)$

(e) none of the above

3. If N is an integer such that 3N is a multiple of 7, then N is

(a) multiple of 21 (b) a multiple of 7

(c) odd (d) even (e) none of the above

4. If x and y satisfy the simultaneous equations

$\frac{x}{2} + \frac{y}{3} = 7$ and $\frac{x}{3} + \frac{y}{4} = 5$, then $x^2 - y^2 =$

(a) 135 (b) 17 (c) -17 (d) -108

(e) none of the above

5. Which one of the following numbers is a perfect square?

(a) 156,250,003 (b) 155,226,682 (c) 154,728,728

(d) 150,160,517 (e) 149,352,841

6. Which one of the following factorizations is incorrect?

(a) $x^4 + x^2y^2 + y^4 = (x^2+xy+y^2)(x^2-xy+y^2)$

(b) $x^5 + x + 1 = (x^2+x+1)(x^3-x^2+1)$

(c) $x^6 + y^6 = (x^2+y^2)(x^4-x^2y^2+y^4)$

(d) $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-2xy-2yz-2zx)$

(e) $x^8 + 4y^8 = (x^4+2x^2y^2+2y^4)(x^4-2x^2y^2+2y^4)$

7. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is 1, then

(a) $a \geq b$ (b) a and b must be of the same sign

(c) a and b must be of opposite sign (d) $c \geq 0$

(e) none of the above

8. The asterisks in the incomplete multiplication below denote missing digits.

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   1*46
x    *5
   6730
 107*8
 114410
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The sum of the three missing digits is (a) 10 (b) 11

(c) 12 (d) 13 (e) none of the above

9. $\text{Log}_8 128 =$

(a) 2 (b) 4 (c) 7/3 (d) an irrational number

(e) none of the above

10. What is the length of a side of the largest cube which can be cut out of a sphere of radius 3 ft?

(a) $3\sqrt{2}$ ft. (b) π ft. (c) $\pi\sqrt{3}$ ft.

(d) $2\sqrt{3}$ ft. (e) none of the above

11. A glass is half full of pure alcohol and another glass twice the volume is one third full. Both glasses are then filled to the top with water and mixed together. What is the fraction of alcohol in the resulting mixture?

(a) 5/6 (b) 5/12 (c) 7/18 (d) 7/12

(e) none of the above

12. Which one of the following statements is false?

(a) In the complex number field, every number has at least one square root.

(b) The product of two rational numbers is a rational number.

(c) The sum of two irrational numbers is an irrational number.

(d) Every non-constant polynomial with complex coefficients has at least one root in the complex number field.

(e) The real cube root of a positive irrational real number is irrational.

13. The equation $4^{x-1} + 4^2 = 5(2^x)$ has

(a) no real solutions (b) exactly one real solution

(c) exactly two real solutions

(d) an infinite number of real solutions

(e) none of the above

14. The equation $\frac{1}{x-3} + \frac{1}{x-2} = \frac{3x-8}{(x-3)(x-2)}$ has

(a) no roots (b) exactly one root (c) exactly two roots

(d) exactly three roots (e) none of the above

15. The expression

$(2\sqrt{3} + \sqrt{10})(7\sqrt{6} - 4)(2\sqrt{3} - \sqrt{10})$ can be simplified to

(a) $14\sqrt{6} - 8$ (b) $7\sqrt{6} - 8$

(c) $a\sqrt{3} + b\sqrt{2} + c\sqrt{30} + d$ where a,b,c, and d are integers.

(d) $a\sqrt{3} + b\sqrt{6} + c\sqrt{10} + d$ where a,b,c, and d are non-zero integers.

(e) none of the above

16. If a and b denote positive integers such that $a < b$, then the proper order of $\sqrt{a} + \sqrt{b}$, $\sqrt{a-1} + \sqrt{b+1}$, and $\sqrt{a+1} + \sqrt{b-1}$ is

(a) $\sqrt{a} + \sqrt{b} \le \sqrt{a-1} + \sqrt{b+1} \le \sqrt{a+1} + \sqrt{b-1}$

(b) $\sqrt{a-1} + \sqrt{b+1} \le \sqrt{a} + \sqrt{b} \le \sqrt{a+1} + \sqrt{b-1}$

(c) $\sqrt{a-1} + \sqrt{b+1} \le \sqrt{a+1} + \sqrt{b-1} \le \sqrt{a} + \sqrt{b}$

(d) $\sqrt{a+1} + \sqrt{b-1} \le \sqrt{a} + \sqrt{b} \le \sqrt{a-1} + \sqrt{b+1}$

(e) $\sqrt{a+1} + \sqrt{b-1} \le \sqrt{a-1} + \sqrt{b+1} \le \sqrt{a} + \sqrt{b}$

17. The remainder when dividing $1 + 2x + 3x^2 + \cdots + nx^{n-1}$ by $(x-1)$ is

(a) 0 (b) 1 (c) n (d) $n(n+1)/2$

(e) none of the above

18. If $n > 1$ is an integer, and if $d(n)$ is the number of positive integers which exactly divide n, then the infinite sequence $n, d(n), d(d(n)), d(d(d(n))) \cdots$

(a) is strictly decreasing

(b) is 1 from some point onwards

(c) is 2 from some point onwards

(d) cannot decrease strictly to the 7th term and then remain constant from there on

(e) none of the above

19. For which angles x between $0°$ and $180°$ is $\frac{1-\sin x}{\cos x} = \frac{\cos x}{1+\sin x}$?

(a) $x = 0°$ only (b) $x = 0°$ and $30°$ only

(c) all angles (d) all angles except $90°$

(e) none of the above

20. Relative to the equation $x^2 = 2y^2$

(a) there is a solution in positive integers

(b) there is a solution in positive rational numbers

(c) there is no solution in rational numbers

(d) if (x_1, y_1) is a solution, and $x_1 \neq 0$, then either x_1 or y_1 is an irrational number.

(e) none of the above

21. The graphs of $y = |x|$ and $y = x^2$ have

(a) exactly one point in common

(b) exactly two points in common

(c) exactly three points in common

(d) exactly four points in common

(e) none of the above

22. How many diagonals are there in a regular polygon of 20 sides?

(a) 340 (b) 176 (c) 170 (d) more than 340

(e) none of the above

23. If a rhombus has side length 1, the sum of the squares of the lengths of the diagonals

(a) $= 4$ (b) < 4

(c) can not be determined from the information given

(d) $= 2\sqrt{2}$ (e) none of the above

24. If A denotes the area of a triangle of base 2 and perimeter 8, then

(a) $0 < A \le 2\sqrt{2}$ (b) $1 < A < 2\sqrt{2}$ (c) $0 < A \le 2$

(d) $2 \le A \le 2\sqrt{2}$ (e) none of the above

25. If x is a real number satisfying the inequality $\big|x + |x-1|\big| + |x| < 3$, then x must satisfy

(a) $0 \le x \le 1$ (b) $x \ge 0$ (c) $0 \le x \le 2/3$

(d) $-1 \le x \le 4/3$ (e) none of the above

26. The expression $\dfrac{1}{1 - \dfrac{1}{2+\sqrt{2}} + \dfrac{1}{2-\sqrt{2}}}$ equals

(a) $\sqrt{2} - 1$ (b) $\sqrt{2} + 1$ (c) $-1 - \sqrt{2}$ (d) $\dfrac{\sqrt{2}}{2}$

(e) none of the above

27. If $\frac{x^{100}}{x^2-3x+2} = P(x) + \frac{ax+b}{x^2-3x+2}$

where $P(x)$ is a polynomial, then the remainder $ax+b$ is

(a) $2^{100}x + 1 - 2^{100}$ (b) $(2^{100}-1)x + 2 - 2^{100}$

(c) $2x - 1$ (d) not determinable from the given data

(e) none of the above

28. Find the diameter of the inscribed circle to a triangle whose sides are 6, 8, and 10 ft. respectively.

(a) 5 ft. (b) $2\sqrt{3}$ ft. (c) 4 ft. (d) 3 ft.

(e) none of the above

29. The number of positive integers ≤ 100 having exactly four (distinct positive integral) divisors is

(a) < 10 (b) > 10 and < 20 (c) 26 (d) 32

(e) none of the above

30. The number of zeros that the number $1001! = 1 \cdot 2 \cdot 3 \cdots 1000 \cdot 1001$ ends in when completely multiplied out is

(a) 100 (b) 110 (c) 113 (d) > 200

(e) none of the above

31. If $F(x,y) = 2x^2 + 3y^2 - 4$, then $F(z,2z) - F(2z,z) =$

(a) $3z^2$ (b) $-3z^2$ (c) $3z^2-4$ (d) $3z^2-8$

(e) none of the above

32. If $\begin{vmatrix} 0 & 1 & \sin x \\ \cos x & 0 & \sin x \\ \cos x & 1 & 0 \end{vmatrix} = 1$, then one solution is given by x =

(a) 90° (b) 30° (c) 60° (d) 45° (e) 75°

33. The sum of the infinite series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$ is

(a) $\frac{3}{4}$ (b) $\frac{7}{8}$ (c) 1 (d) $\frac{3}{2}$

(e) none of the above

34. Each face of a cube is colored either red or blue. How many distinct colorings of the cube are possible independent of orientation? (For example, there is only one way to color the cube such that one face is red and the remaining faces are blue).

(a) 6 (b) 8 (c) 10 (d) 12

(e) none of the above

35. The real solution set for the inequality $|x+1| < |x| + 2$ is

(a) $x < 0$ (b) $x \geq 0$ (c) $x \leq 10$ (d) all real x

(e) none of the above

36. The minimum degree of a polynomial with integer coefficients having $\sqrt{3} + \sqrt{2} + i$ as a root is

(a) 3 (b) 6 (c) 8 (d) 9

(e) none of the above

37. (m,n) and $[m,n]$ denote the greatest common divisor and least common multiple of the integers m,n, respectively. If for three positive integers a,b,c,

$$(a,b) = 6, \qquad [a,b] = 180$$
$$(b,c) = 30, \qquad [b,c] = 90,$$
$$(c,a) = 6, \qquad [c,a] = 60,$$

then abc

(a) is $< 6\cdot6\cdot30$ (b) is $> 60\cdot90\cdot180$ (c) $= 32{,}400$

(d) cannot be determined from the given data

(e) none of the above

38. Let $y = F(x)$ be a real-valued function of x (real). If $x^2F(x) + 2x^2 \equiv 8 + F(x)$, what interval is excluded as possible values of $F(x)$?

(a) $-8 \le y \le -2$ (b) $-8 < y \le -2$ (c) no interval

(d) $y > 0$ (e) none of the above

39. If $P = (1 + 3^{-1})\ (1 + 3^{-2})\ (1 + 3^{-4})\ (1 + 3^{-8})\cdots(1 + 3^{-(2^n)})$, then for $n > 10{,}000$

(a) $\frac{3}{2} > P > \frac{4}{3}$ (b) $2 > P > \frac{3}{2}$ (c) $4 > P > 2$

(d) $5 > P > 4$ (e) $P > 5$

40. A three digit number in base 7 equals the reversed three digit number in base 9. The sum of the three digits in base 9

(a) is not uniquely determined (b) $= 8$ (c) $= 11$

(d) $= 16$ (e) none of the above

The following Michigan companies and professional organizations have made contributions to the scholarship fund for this competition.

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