SOLUTIONS TO PART II

ELEVENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

(The solutions to problems 2,3 and 4 are verbatim solutions presented by students)

Comment: No contestant really handled the sign combinations adequately. Many solved the problem correctly up to the question of signs and were given essentially full credit.

1. $(x + y)(x + z) = a^2$ 2. $(x + y)(y + z) = b^2$

3. $(x + z)(y + z) = c^2$

$$\frac{(1) \times (2)}{(3)} \Rightarrow (x + y)^2 = \frac{a^2 b^2}{c^2} \Rightarrow x + y = \frac{ab}{c} \text{ or } x + y = -\frac{ab}{c}$$

Case I $\quad x + y = \frac{ab}{c} \Rightarrow (\frac{ab}{c})(x + z) = a^2 \Rightarrow x + z = \frac{ac}{b}$

$$\Rightarrow (x + y) - (x + z) = y - z = \frac{ab}{c} - \frac{ac}{b}$$

$$(x + z)(y + z) = (\frac{ac}{b})(y + z) = c^2 \Rightarrow y + z = \frac{bc}{a}$$

$$(y + z) + (y - z) = \frac{bc}{a} + \frac{ab}{c} - \frac{ac}{b} \Rightarrow y = \frac{1}{2}\left[\frac{bc}{a} + \frac{ab}{c} - \frac{ac}{b}\right]$$

$$x + y = \frac{ab}{c} \Rightarrow x = \frac{ab}{c} - \frac{1}{2}\left[\frac{bc}{a} + \frac{ab}{c} - \frac{ac}{b}\right] = \frac{1}{2}\left[\frac{ab}{c} + \frac{ac}{b} - \frac{bc}{a}\right]$$

$$x + z = \frac{ac}{b} \Rightarrow z = \frac{ac}{b} - \frac{1}{2}\left[\frac{ab}{c} + \frac{ac}{b} - \frac{bc}{a}\right] = \frac{1}{2}\left[\frac{ac}{b} + \frac{bc}{a} - \frac{ab}{c}\right]$$

Case II $\quad x + y = -\frac{ab}{c} \Rightarrow (-\frac{ab}{c})(x + z) = a^2 \Rightarrow x + z = -\frac{ac}{b}$

$$\Rightarrow (x + y) - (x + z) = y - z = -\frac{ab}{c} + \frac{ac}{b}$$

$$(x + z)(y + z) = (-\frac{ac}{b})(y + z) = c^2 \Rightarrow y + z = -\frac{bc}{a}$$

$$(y + z) + (y - z) = -\frac{bc}{a} + (-\frac{ab}{c} + \frac{ac}{b}) \Rightarrow y = \frac{1}{2}\left[-\frac{bc}{a} - \frac{ab}{c} + \frac{ac}{b}\right]$$

$$x + y = -\frac{ab}{c} \Rightarrow x = -\frac{ab}{c} - \frac{1}{2}\left[-\frac{bc}{a} - \frac{ab}{c} + \frac{ac}{b}\right] = \frac{1}{2}\left[-\frac{ab}{c} + \frac{bc}{a} - \frac{ac}{b}\right]$$

$$x + z = -\frac{ac}{b} \Rightarrow z = \frac{1}{2}\left[\frac{ab}{c} - \frac{bc}{a} - \frac{ac}{b}\right]$$

II

Verbatim solution by Robert Rozenberg - Oak Park High School

Consider ΔPQR. By the cosine law

$\overline{PQ}^2 + \overline{QR}^2 - 2\,\overline{PQ} \cdot \overline{QR} \cos \angle PQR = \overline{PR}^2$ So

$25 + 13 - 2 \cdot 5 \cdot \sqrt{13} \cos \angle PQR = 36$ So

$\cos \angle PQR = \dfrac{1}{5\sqrt{13}}$. However $\sin^2 + \cos^2 = 1 = \sin^2 + \dfrac{1}{325}$

So $\sin \angle PQR = \sqrt{\dfrac{324}{325}} = \dfrac{18}{5\sqrt{13}}$. Thus the area of ΔPQR

$= \frac{1}{2}\,\overline{PQ} \cdot \overline{QR} \sin \angle PQR = 9$.

$m \angle BQC = 180 - m \angle PQR$, So $\sin \angle BQC = \sin \angle PQR = \dfrac{18}{5\sqrt{13}}$

So the area of $\Delta BQC = \frac{1}{2}\,\overline{BQ} \cdot \overline{QC} \sin \angle BQC = 9$

Furthermore, $\sin \angle ERD = \sin \angle QRP$ since $m \angle ERD =$

$180 - m \angle QRP$ so area of $\Delta ERD = \frac{1}{2}\,\overline{ER} \cdot \overline{DR} \cdot \sin \angle ERD$

$= \frac{1}{2}\,\overline{PR} \cdot \overline{QR} \sin \angle PRQ =$ area of $\Delta PQR = 9$.

Similarly $\sin \angle APF = \sin \angle QPR$ so

Area of $\Delta APF = \frac{1}{2}\,\overline{AP} \cdot \overline{PF} \cdot \sin \angle APF = \frac{1}{2}\,\overline{PQ} \cdot \overline{PQ} \sin \angle QPR$

$=$ area of $\Delta PQR = 9$.

Hence, area of hexagon ABCDEF = area of ΔAPF

+ area of □ PFER + area of ΔERD + area of □ QRDC

+ area of ΔBQC + area of □ ABQR + area of ΔPQR

$= 9 + 36 + 9 + 13 + 9 + 25 + 9 = 110$ sq. units .

III

Verbatim solution by Ken Rietz - Dearborn High School

$\frac{P}{p} + \frac{Q}{q} + \frac{R}{r} = K$, some integer

$$\frac{Pqr + Qpr + Rpq}{Pqr} = K$$

$$P\,q\,r + Q\,p\,r + R\,p\,q = K\,p\,q\,r$$

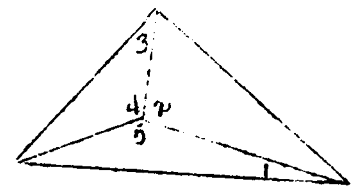
$$P\,q\,r + Q\,p\,r = K\,p\,q\,r - R\,p\,q$$

$$r(Pq + Qp) = pq(Kr - R)$$

But since p, q and r have no common factors larger than 1, r must be a factor of Kr - R . But r is obviously a factor of Kr and so must also be a factor of R or -R . Therefore, $\frac{R}{r}$ is an integer. Similarly $\frac{P}{p}$ and $\frac{Q}{q}$ are integers.

IV

Verbatim solution by John Walbridge - Escabana High School



All 4 faces of the tetrahedron are congruent because their sides are ≅ $\therefore \angle 1 = \angle 2 = \angle 3$. Thus the sum of the angles at any vertex = 180° since they are formed of angles of a Δ .

Assume 2 obtuse .

$\therefore$ 4 and 5 are acute

since $4 + 5 + 2 = 180$, $4 + 5 < 90°$

However, the sum of any two angles at the vertex of a 3 dimensional figure must be greater than the third. Therefore 2 cannot be obtuse.

The same argument shows it cannot be 90° . Hence, 2 is acute and similarly so are 4 and 5 .

V

<u>Comment</u> - no student presented a complete solution to the problem.

Let P be the origin of a rectangular coordinate system and let $(x_1 y_1)$, $(x_2 y_2)$, (x_3, y_3), $(x_4 y_4)$ be the coordinates of P_1 , P_2 , P_3 , P_4 respectively.

Then $\overline{PP_1}^2 + \overline{PP_2}^2 + \overline{PP_3}^2 + \overline{PP_4}^2 = \sum_{i=1}^{4}(x_i^2 + y_i^2) = \sum_{i=1}^{4} x_i^2 + \sum_{i=1}^{4} y_i^2$

Since $\overline{P_i P_j} \geq 2 \Rightarrow (x_i - x_j)^2 + (y_i - y_j)^2 \geq 4$

Adding these six inequalities we get

$3 \Sigma x_i^2 + 3 \Sigma y_i^2 - 2 \Sigma x_i x_j - 2 \Sigma y_i y_j \geq 24$

Regrouping and adding $\Sigma x_i^2 + \Sigma y_i^2$ to both sides gives

$4\Sigma x_i^2 + 4\Sigma y_i^2 \geq \Sigma x_i^2 + \Sigma y_i^2 + 2\Sigma x_i x_j + 2\Sigma y_i y_j + 24$

Noting that $(\Sigma x_i)^2 = \Sigma x_i^2 + 2\Sigma x_i x_j$ and $(\Sigma y_i)^2 = \Sigma y_i^2 + 2\Sigma y_i y_j$

it follows that

$\overline{PP_1}^2 + \overline{PP_2}^2 + \overline{PP_3}^2 + \overline{PP_4}^2 = \Sigma x_i^2 + \Sigma y_i^2 \geq \frac{1}{4}(\Sigma x_i)^2 + \frac{1}{4}(\Sigma y_i)^2 + 6$

which is greater than or equal to 6 regardless of the values of x_i and y_i .