ELEVENTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America, Michigan Colleges and Universities, Professional Organizations, and Industries.

PART I

OCTOBER 25, 1967

INSTRUCTIONS

(to be read aloud to class by supervisor or proctor)

1. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

2. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out ideas on scratch paper before selecting the answer.

3. Your score on the test will be the number right. You are advised to guess an answer in those cases where you cannot determine the right answer but are able to eliminate some of the alternatives as impossible.

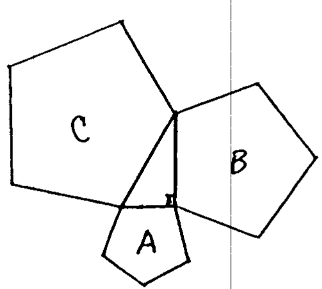
4. Usually a score of 15 or more will allow you to become a finalist and write the second exam. To improve your score, be careful to complete all problems which you can do successfully before working on the other problems. Several questions, distributed throughout the test, can be answered by using only the first two years of high school mathematics.

5. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Do not make calculations on the answer sheet.

6. In each of the questions five different possible responses are proposed. In many cases the fifth alternative is listed "E none". In such cases if you believe none of the first four alternatives to be correct, mark E.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of the competition. If you have questions concerning the instructions ask them now.

7. Regular pentagons are constructed on the sides of a right triangle as illustrated. If A,B, and C denote the respective areas of the pentagons, then



A) $A + B > C$

B) $A + B = C$

C) $A + B < C$

D) $A^2 + B^2 = C^2$

E) none

8. Two sides of a triangle are 5 and 10 units long respectively. The area of the triangle

A) can be greater than 30 sq. units

B) is less than or equal to 25 sq. units

C) is greater than 25 sq. units

D) must be greater than 1 sq. unit

E) none

9. The value of $(\cos \frac{\pi}{12} + \sin \frac{\pi}{12})^2$ is

A) $\frac{3}{2}$

B) 1

C) $1 + \frac{\sqrt{3}}{2}$

D) $\frac{2}{3}$

E) none

10. Three dice are thrown, the probability that the sum of the readings of the top faces is 5 is

A) $\frac{1}{12}$

B) $\frac{1}{18}$

C) $\frac{1}{36}$

D) $\frac{1}{108}$

E) none

11. If $x^5 + 2x^3 + ax^2 + b$ is divisible by $x^3 + 1$ then

A) $a = 0,\ b = 3$

B) $a = 2,\ b = 1$

C) $a = 1,\ b = 2$

D) $a + b = 4$

E) none

16. Using an unmarked straight edge and compasses in accordance with prescribed euclidean rules it is impossible to construct

A) a regular pentagon

B) a straight line segment of length $\sqrt{1+\sqrt{3}}$ if a straight line segment one unit long is given

C) a regular polygon of 7 sides

D) a circle with length equal to the sum of the lengths of two given circles

E) a regular polygon of 17 sides

17. How many real roots does $\frac{1}{x} = \sqrt{x^2 - 2}$ have?

A) 1

B) 2

C) 3

D) 4

E) none

18. Which of the following statements about polynomials in a single variable x, of degree n, and with real coefficients is true?

A) They are always factorable into n linear factors, each with real coefficients

B) (A) is false but such a polynomial may always be factored into a product of linear and quadratic factors, each having real coefficients

C) If n is odd such a polynomial has either no non-real roots or an odd number of non-real roots.

D) If one root of such a polynomial is non-real, then the sum of the roots must be non-real.

E) none

19. If $n = (1 - \frac{1}{4})(1 - \frac{1}{9}) \cdots (1 - \frac{1}{k^2}) \cdots (1 - \frac{1}{10,000})$, then

A) $\frac{1}{4} < n < \frac{1}{2}$

B) $n < \frac{1}{4}$

C) $n > \frac{11}{20}$

D) $n = \frac{101}{200}$

E) none

20. The equation $x^7 + 4x^6 - x^5 + x^2 - 2x - 3 = 0$ has

A) all rational roots

B) -3 as the product of all of its roots

C) -2 as the sum of all of its roots

D) no more than 3 positive roots

E) none

26. The positive integral solutions of $(x+2)! - (x+1)! - x! = x^4 + x^2$ are such that

A) $\sin(\frac{\pi}{x})$ is $\frac{1}{2}$

B) $x = 3$

C) $x = 3$ and $x = 8$

D) $x = 3$, 5 and 8

E) none

27. If in ΔABC $\overline{AB} = 6$, $\overline{BC} = 4$, $\overline{AC} = 5$ and the bisector of $\angle B$ meets AC in P. Then

A) $\overline{CP}$ is $\frac{5}{2}$

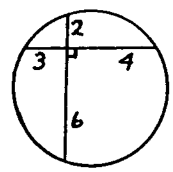
B) $\overline{CP}$ cannot be determined from the given data

C) $\overline{CP} = 3$

D) $\overline{CP} = 2$

E) none

28. Two perpendicular chords of a circle are as shown. The diameter of the circle



A) is $\sqrt{67}$

B) is 8

C) is $\sqrt{65}$

D) cannot be determined from the given data

E) none

29. In a certain boys school with more than one student, it is found that, if one boy is left out, the remaining boys, each playing in every sport, can form an even (not odd) number of football teams (11 members) in the fall, an even number of basketball teams (5 members) in the winter, and an even number of baseball teams (9 members) in the spring. The minimum number of boys in the school is

A) Between 300 and 499

B) Between 500 and 699

C) Between 700 and 899

D) Greater than 900

E) none

30. The intersection of a cube and a plane can never be

A) an equilateral triangle

B) a rhombus

C) a regular hexagon

D) a right triangle

E) none