NINTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America,
Michigan Colleges and Universities, Professional Organizations, and Industries.

PART I

NOVEMBER 11, 1965

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

2. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out your ideas on scratch paper before selecting the answer.

3. Your score on the test will be the number right. You are advised to guess an answer in those cases where you cannot determine the right answer but are able to eliminate some of the alternatives as impossible.

4. The average participant will have less than ten correct answers. To improve your score, be careful to complete all problems which you can do successfully before working on the other problems. Several questions, distributed throughout the test, can be answered by using only the first two years of high school mathematics.

5. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Do not make calculations on the answer sheet.

6. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of this competition. If you have questions concerning the instructions, ask them now.

PART I

1. Two numbers whose sum is 4 and difference is 6 are roots of

   (A) $x^2 - 4x + 5 = 0$

   (B) $x^2 + 4x - 5 = 0$

   (C) $x^2 - 4x - 5 = 0$

   (D) $x^2 - 4x + 6 = 0$

   (E) None of the above

2. Each number in a set of ten numbers is decreased by the average A of the ten numbers. Then the average of these ten is

   (F) $\frac{9}{10}$ (G) $\frac{1}{2}$ A

   (H) -9A (I) $\frac{9}{10}$ A

   (J) 0

3. Simplify

   $$(\sqrt[5]{\sqrt[10]{a^4}})^{12} \cdot (\sqrt[10]{\sqrt[5]{a^4}})^{13}$$

   (A) a^{14} (B) a^2

   (C) $a^{\frac{1}{2}}$ (D) a

   (E) a^4

4. In base four, one counts: 1, 2, 3, 10, 11, 12, ··· . The base four symbol for the thirty-sixth number is:

   (F) 36 (G) 310

   (H) 204 (I) 90

   (J) 210

5. The repeating decimal 3.0666 ··· is

   (A) $\frac{46}{15}$ (B) $3\frac{2}{3}$

   (C) 3.07 (D) $\frac{37}{12}$

   (E) None of the above

6. (Use the approximations log 2 = .3010 and log 3 = .4771.) The number of digits in 18^{10} is

   (F) 19 (G) 13

   (H) 12 (I) 11

   (J) None of the above

7. If the constant term of $(x + \frac{1}{x})^n$ is 20, then n equals

   (A) 6

   (B) -3

   (C) 8

   (D) 5

   (E) 20

8. If $a^2 + \frac{1}{a^2} = 1$, then $a^3 + \frac{1}{a^3} =$

   (F) $\sqrt{2}$

   (G) 1

   (H) 0

   (I) 2

   (J) $6\sqrt{3}$

9. The four roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$ are i, $-i$, $1 + i$, and $1 - i$. If $a = 1$, then

(A) $b = -2$ (B) $b = -1$

(C) $b = 0$ (D) $b = 1$

(E) None of the above

10. If $f(z) = z^2 - 9$ and $z \neq 0$, then $f(z + x) = 0$ when

(F) $x = 3$

(G) $x = -3$

(H) $x = 3 + z$

(I) $x = 3 - z$

(J) None of the above

11.
$$\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0$$

The solutions to this equation are

(A) $x = a$, $x = b$

(B) $x = ma$, $x = mb$

(C) $x = \frac{a}{b}$, $x = -\frac{a}{b}$

(D) $x = \frac{a}{m}$, $x = \frac{b}{m}$

(E) None of the above

12. The maximum number of regions that can be formed on the surface of the sphere by three great circles is

(F) 14 (G) 6

(H) 7 (I) 16

(J) 8

13. If a polygon has 35 diagonals, then the number of sides is

(A) 10 (B) 35

(C) 8 (D) 14

(E) None of the above

14. Let s_1, s_2, s_3 be respectively the sum of the first n terms, 2n terms, and 3n terms of the arithmetic progression a, $a + d$, $a + 2d$, $\cdots$. Then $s_3 - 3(s_2 - s_1)$ depends upon

(F) a, d, and n

(G) a and d alone

(H) none of a, d, n

(I) n and d alone

(J) a alone

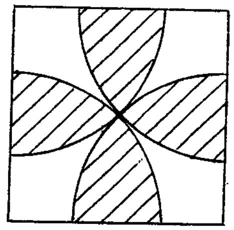
15. If $\log_x(\frac{27}{64}) = -3$, then $x =$

(A) $-\frac{3}{4}$ (B) $\frac{3}{4}$

(C) $\frac{4}{3}$ (D) 10

(E) 3

16.



The side of the square has length 1. The arcs are centered on the vertices and drawn through the center of the square. The area of the shaded region is

(F) $\frac{\pi}{2} - 1$ (G) $\frac{\pi}{8} - \frac{1}{8}$

(H) $2 - \frac{\pi}{2}$ (I) $\frac{1}{2} - \frac{\pi}{8}$

(J) None of the above

17. X can do a job in three days. If X and Y together can do the job in two days, and Y works twice as rapidly as Z, then X and Z together can do the job in

(A) 4 days (B) $2\frac{2}{5}$ days

(C) $3\frac{1}{2}$ days (D) $2\frac{1}{2}$ days

(E) 1 day

18. A can circle a track in 60 seconds, B in 65 seconds. How many minutes must A run in order to lap B?

(F) 20 (G) 12

(H) 21 (I) 13

(J) None of the above

19. How many minutes after four o'clock is the hour hand 180° from the minute hand?

(A) $54\frac{1}{2}$ (B) 55

(C) $54\frac{3}{11}$ (D) $54\frac{1}{6}$

(E) $54\frac{6}{11}$

20. If $\frac{2x + 7}{2x^2 - x - 1} = \frac{A}{2x + 1} + \frac{B}{x - 1}$, then

(F) A = 7 and B = 2

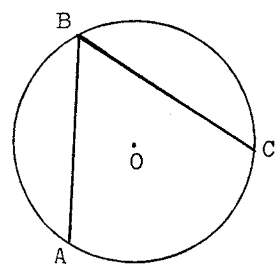
(G) A = 3 and B = -4

(H) A = 2 and B = 7

(I) A = -4 and B = 3

(J) None of the above

21.



AB = BC = 12. ∠ABC = 60°. The radius of circle O is

(A) 6 (B) $12\sqrt{3}$

(C) $4\sqrt{3}$ (D) $4\sqrt{2}$

(E) $6\sqrt{3}$

22. Which of the numbers below is closest to $\frac{1}{7}\left(\frac{3}{5}+\frac{6}{7}\right)^{-1}$?

(F) $\frac{1}{9}$ (G) $\frac{1}{10}$

(H) $\frac{1}{11}$ (I) $\frac{1}{12}$

(J) $\frac{2}{21}$

23. Point P is external to a square of side s. Then the locus of midpoints of all segments from P to the square is

(A) 2 perpendicular segments

(B) 4 parallel segments

(C) a square of side s

(D) a square of side $\frac{s}{2}$

(E) a circle

24. An equation for a line passing through the origin and perpendicular to the line given by $x - 3y = 2$ is

(F) $x + 3y = 0$ (G) $x - 3y = 0$

(H) $3x + y = 0$ (I) $3x - y = 0$

(J) $x - 3y = 2$

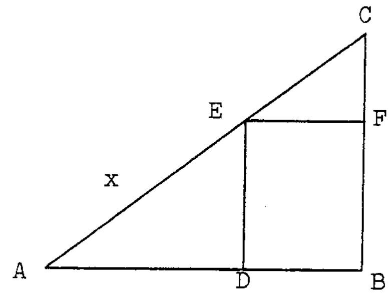
25. A group of 5 boys and 5 girls are to choose a committee of 3 boys and 2 girls from their group. In how many ways can they do this?

(A) 20 (B) 100

(C) 1200 (D) 80

(E) 120

26.



In right triangle ABC, AB = 4, BC = 3, ED⊥AB, EF⊥BC, AE = x. Then the area of rectangle DEFB is

(F) 3

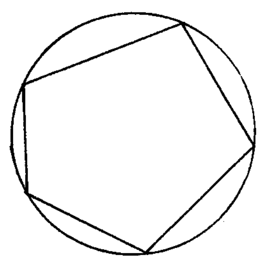
(G) $\frac{12}{5}\left(x - \frac{x^2}{5}\right)$

(H) $6 - \frac{24}{25}x^2$

(I) $\frac{7}{5}\left(x - \frac{x^2}{5}\right)$

(J) None of the above

27.



A five-sided polygon is inscribed in a circle. An angle is inscribed in each arc cut off by the sides. Then the sum of these five angles is

(A) 1440° (B) 540°

(C) 720° (D) 900°

(E) 360°

28. The equations $x - y + z = 1$, $x + y - z = 1$, and $y = z$ have

(F) no common solutions

(G) exactly one common solution

(H) exactly three common solutions

(I) an infinite number of common solutions

(J) a finite number of common solutions, but not 0, 1, or 3.

29. The parabola $y = x^2 - 2$ and the ellipse $x^2 + 4y^2 = 16$ have

(A) no points in common

(B) exactly one point in common

(C) exactly two points in common

(D) exactly three points in common

(E) exactly four points in common

30. A circle is cut into sectors of 90° and 270° and each sector is shaped into a cone. A ratio of the heights of these cones is

(F) $\sqrt{15} : \sqrt{7}$ (G) $15 : 7$

(H) $3 : 1$ (I) $\sqrt{3} : 1$

(J) None of the above

31. If $x > y$, $|x - y|$ never exceeds

(A) either x or y

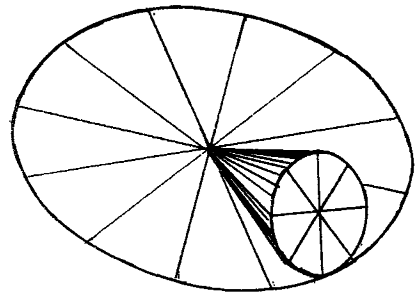
(B) $x + y$

(C) $x - y$

(D) $|x| - |y|$

(E) $|x + y|$

32.



A right circular cone with altitude h and base radius r rolls on a plane with vertex fixed. If the cone's surface is covered three times as it rolls in a complete circle, then the ratio of h to r is

(F) $4\sqrt{2}$ (G) $\sqrt{2}$

(H) $2\sqrt{2}$ (I) 3

(J) 2.5

33. The solution set for the inequality $\frac{x - 2}{x + 2} > 1$ is given by

(A) empty set

(B) $x < 0$ and $x \neq -2$

(C) $x < 0$

(D) $-2 < x < 2$

(E) $x < -2$

34. The coefficient of $x^2y^2z^3$ in the expansion of $(x + y + z)^7$ is

(F) 360 (G) 84

(H) 7 (I) 210

(J) 280

35. Consider the two numbers $(2\cdot3\cdot5\cdots p)$ and $(2\cdot3\cdot5\cdots p) + p$ where the product includes all primes up to the prime p. If $\pi(p)$ is the number of primes less than or equal to p, then the number of primes between these two integers is

(A) sometimes more than one

(B) always 0 (C) $\pi(p)$

(D) $p - \pi(p)$ (E) at most 1

36. There are no points (x,y) satisfying both $y > 3x + 1$ and $y < x^2$, in quadrants

(F) I or IV (G) IV only

(H) III or IV (I) I, III, or IV

(J) II or III

37. The largest integer which will divide $7^{2n+1} + 1$ for all integers $n \geqslant 0$ is

(A) 2 (B) 4

(C) 8 (D) 16

(E) 1

38. $x = \pm\sqrt{r^2 - t}$

$y = \pm\sqrt{t}$ $\qquad 0 \leqslant t \leqslant r^2$

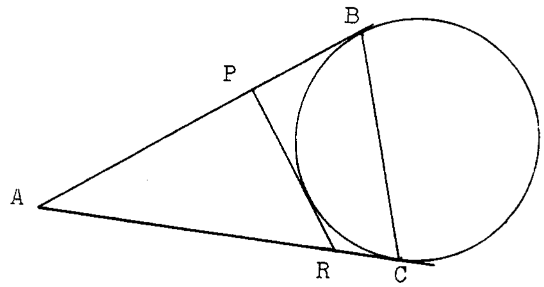
The set of all points (x,y) which satisfy these equations is

(F) square (G) line

(H) ellipse (I) parabola

(J) circle

39.



Lines AB, AC, and PR are tangent to the circle. If the perimeter of triangle APR = 30 and BC = 10, then the perimeter of triangle ABC is

(A) 60 (B) 50

(C) 45 (D) 40

(E) impossible to determine with given information

40. Assume that the following three statements are true:

1) All cats with green eyes like milk.

2) Not all cats with long tails have green eyes.

3) No cat with a short tail likes milk.

A valid conclusion that can be drawn is

(F) No long-tailed cat has green eyes.

(G) Some long-tailed cats do not like milk.

(H) Some cats who do not like milk have long tails.

(I) Some cats with short tails have green eyes.

(J) None of the above.

The following Michigan companies and professional organizations have made contributions to the scholarship fund for this year's competition.

- Aeroquip Corporation, Jackson
- Burroughs Corporation, Detroit
- Clark Equipment Company, Battle Creek
- Consumers Power Company, Jackson
- The Michigan Council of Teachers of Mathematics
- Packaging Corporation of America, Filer City
- Thompson Ramo Wooldridge, Incorporated, Warren
- Whitman and Barnes, Plymouth

The names of other companies, contributing to the scholarship fund during the next few months, will be reported in later announcements.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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