EIGHTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

December 10, 1964

Part II Solutions

Please see examination booklets for a statement of the problems. The solutions given below were constructed prior to correcting the part II booklets. Usually the participants devise several different approaches and discover solutions which are more elegant.

1. The key idea is that all tangents to a sphere from a given point have the same length.
   Label the four vertices of the tetrahedron A, B, C, and D. Let the length of the segment between A and the point of tangency on AB be denoted by a_b. Similarly, b_a denotes the length of the segment from B to the point of tangency on AB. Note that $a_b + b_a = |AB|$.

$$\begin{array}{ccccc} a_b & = & a_c & = & a_d \\ b_a & = & b_d & = & b_c \\ c_d & = & c_a & = & c_b \\ d_c & = & d_b & = & d_a \\ \hline \end{array}$$

$$|AB| + |BC| = |AC| + |BD| = |AD| + |BC|$$

2. By computing the product of the first 2, 4, 6, $\cdots$ factors, we can construct the following table:

n	
2	$3/4$
3	$3/4 \cdot 8/9 = 2/3$
4	$3/4 \cdot 8/9 \cdot 15/16 = 5/8$
5	$\cdots\cdot = 3/5$
6	$\cdots\cdot = 7/12$
7	$\cdots\cdot = 4/7$

It is tempting to adjust the fractions in the second column so their denominators are all equal to 2n. When this is done, a pattern is discovered which suggest the answer $\frac{n+1}{2n}$. This can be proved by mathematical induction.

Alternately, split the product into two pieces $(1 + 1/2)(1 + 1/3) \cdots (1 + 1/n)$ and $(1 - 1/2)(1 - 1/3) \cdots (1 - 1/n)$. The first expression equals $\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{n+1}{n} = \frac{n+1}{n}$ and the second expression equals $\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-1}{n} = \frac{1}{n}$

The answer is $\frac{n+1}{2}\cdot\frac{1}{n} = \frac{n+1}{2n}$.

3. $(x + 1) + \frac{1}{x+1} + x + 4 + \frac{4}{x+4} = x + 2 + \frac{2}{x+2} + x + 3 + \frac{3}{x+3}$

$$\frac{1}{x+1} + \frac{4}{x+4} = \frac{2}{x+2} + \frac{3}{x+3}$$

etc

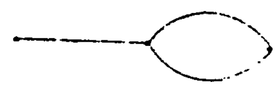
$$x(2x + 5) = 0$$

$$x = 0 \text{ or } x = -5/2$$

4. We need to show that $\angle BAH = \angle DHC$. Because of the arcs subtended, $\angle BAH = \angle DBC$. If a circle is constructed using BC as a diameter, the circle will pass through D and H. In this circle $\angle DBC$ and $\angle DHC$ subtend the same arc.

5. At each corner a decision must be made to go either right (R) or left (L), unless one has already gone right eight times or left seven times. Each path can be described by a sequence of R's and L's containing eight R's and seven L's. Two different permutations describe different paths. The total number of paths is determined by counting the number of ways we can select from fifteen positions the eight positions for the R's. The answer is $\binom{15}{8}$. Alternately, one can start at A and count the number of paths to each corner near A. This strategy of changing the problem to a much easier problem (i.e. imagining B nearer to A) is often very effective in helping find a solution. In this case a pattern quickly becomes evident which permits one to compute the final answer.

6. The problem as stated is false. The following graph is a counter-example.



The problem should read "Prove that if G is a graph with two or more vertices in which there is at most one edge connecting any pair of vertices and no edge connects a vertex to itself then G is not heterogeneous."

<u>Proof</u> Assume that G is heterogeneous

Let the number of vertices in G be n and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the orders of the n vertices. Each vertex can have at most n - 1 edges because of the conditions stated in the problem. Thus $0 \leqslant \alpha_i \leqslant n - 1$. But there are exactly n possible values for α_i between 0 & n - 1, inclusive. If G is heterogeneous, there is an α_r which equals 0. This means that no edges are attached to the r^{th} vertex and all other α's must be $\leqslant n - 2$. However there are (n - 1) α's other than α_r and only n - 2 choices between 1 and n - 2, inclusive, for these α's. Therefore two α's are equal and G is not heterogeneous.