EIGHTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

November 5, 1964

Hints or Suggestions for Part I

1. Each side is of length 6 so altitude is $3\sqrt{3}$ and the area is $9\sqrt{3}$.

2. Each central angle is 20° and $360/20 = 18$.

3. This is a standard problem in inequalities.

4. If $x > 0$, the given inequality is equivalent to $x > 1$; if $x < 0$, the inequality is always true. The solution set is $\{x \mid x < 0 \text{ or } x > 1\}$ and the answer is (H).

5. If "p" is a statement and "$\sim$p" its negative, then "p or $\sim$p" is always true and also "p and $\sim$p" is always false. Alternately, we might note that (6) says that the set M of all men is contained in the set G of all good cooks. To deny that M is contained in G only means that at least one element in M is not in G.

6. $x^4 - 2x^2y^2 + y^4 = (x - y)^2(x + y)^2$. It is easily seen that $x^2 - y^2$, $x^2 - 2xy + y^2$, and $x + y$ divide the expression. Since $x^3 - xy^2 + yx^2 - y^3 = (x^2 - y^2)(x + y)$, the answer is (J).

7. Unless one can divide in base 3, it might be better to convert all numbers to base 10, solve, and convert back to base 3. Division in base 3 is as follows:

$$\begin{array}{r} 110 \\ 21\overline{)10010} \\ \underline{21} \\ 21 \\ \underline{21} \\ 0 \end{array}$$

8. Let x_1, x_2 be the discarded numbers and $x_3, \cdots, x_{60}$ the remaining numbers, then the given information leads to two equations:

$$x_1 + x_2 + \cdots + x_{60} = 60(24)$$

$$x_3 + \cdots + x_{60} = 58(25)$$

9. If the quadratic formula is used, one needs to simplify $\sqrt{8 + 6i}$. Alternately, it is tempting to try -2 as a solution. Since it works, one checks 1 + i directly or by using the relationships between the roots and the coefficients.

10. $x^2 - 2xy + y^2 = 3$, $x^2 + 2xy + y^2 = 4 \Longrightarrow 4xy = 1$.

11. This was an easy problem.

12. List all solutions to $x + y = 15$ and check if xy is a perfect square.

13. Consider the form of the equation for each of the four cases where $x - 2$ is positive or negative and $y + 3$ is positive or negative.

14. A simple problem.

15. Using the sum of the angles in ΔAPB, we see that $\angle APB$ is 105°.

16. Just use distance = rate x time and keep units consistent.

17. Subtract $x = .327327 \cdots$ from $1000x = 327.327327 \cdots$.

18. If two triangles have equal bases and equal altitudes, then their areas are equal. This proves (G) is the answer. The other alternatives can be eliminated by suitable figures.

19. This is an easy problem.

20. A standard problem from geometry. Apply pythagorean theorem to triangle CC'X where X is on C'T' and on the line through C and parallel to TT'.

21. Draw lines from the center of the small circle to the points of contact with the two diameters and the large circle. These segments are all equal to the radius x of the small circle and $(1 - x)^2 = x^2 + x^2$.

22. $(10t + u) + (10u + t) = 11(t + u)$. Answer is (I).

23. Total number of permutations minus those where the two particular people sit together = $4! - 2 \cdot 3!$

24. Solve for y in terms of x by quadratic formula. The discriminant is $x^2 + 4x$ and y will be real if and only if $x^2 + 4x > 0$.

26. After one substitution, the result is x. The final expression is $\frac{x + 1}{x - 1}$.

27. The problem should read "The number of <u>positive</u> integers . . ." From the set of positive integers less than 1,000, let A be the set of squares, let B be the even numbers, and let C be those divisible by five. If No(S) is the number of elements in the set S, then the answer is $999 - No(A \cup B \cup C)$, which is computed by using the formula $No(A \cup B \cup C) = No(A) + No(B) + No(C) - No(A \cap B) - No(A \cap C) - No(B \cap C) + No(A \cap B \cap C)$. This last formula is easily derived by inspection of the Venn diagram for three subsets.

28. Either use the formula $\log_b N = \frac{\log_a N}{\log_a 6}$ of set $\log_5 9 = x$ and translate into exponential form ($5^x = 9$) and then take logarithms of both sides, using logarithms to base 10.

29. Straight forward substitution.

30. Set $x^{10} - 1 = y^5 - 1$ where $y = x^2$ and use the remainder theorem. The remainder when $y^5 - 1$ is divided by $(y - a)$ is $a^5 - 1$.

31. $\frac{x + y}{2} = \sqrt{2}\sqrt{xy}$. Write as a quadratic in y/x.

32. The given equation is equivalent to $10^{(x^2 + x)} - 10^2 = 0$.

33. A network can be traversed if and only if it contains 0 or 2 vertices with an odd number of edges. This permits one to show that no more than nine of the edges can be traversed.

34. $\frac{-b + \sqrt{D}}{2a} = 3\,\frac{-b - \sqrt{D}}{2a} \Longrightarrow 3b^2 = 16\,ac.$

35. Use the relationship between the roots and the coefficients.
$r + s = -b/a, \; rs = c/a$
$\frac{1}{r} + \frac{1}{s} = -p$

36. $[\pi(6 + x)^2 \cdot 4 - \pi(6)^2 \cdot 4] = 2[\pi(6)^2(4 + x) - \pi(6)^2 \cdot 4].$

37. $\cos(3\theta) = \cos(\theta + 2\theta) = \cos\theta\cos 2\theta - \sin\theta\sin 2\theta = \cdots$
$= 4\cos^3\theta - 3\cos\theta$
$\cos\theta + \cos 3\theta = 4\cos^3\theta - 2\cos\theta$
$= 2\cos\theta[2\cos^2\theta - 1]$
$= 2\cos\theta\cos 2\theta$

$$\frac{\cos\theta + \cos 3\theta}{\cos^2\theta - \sin^2\theta} = 2\cos\theta$$

Alternately, and easier, use the identity

$$\cos A + \cos B = 2\cos\frac{A + B}{2}\cos\frac{A - B}{2}.$$

38. By pythagorean theorem, AC = 10. If the answer is y,

$$\frac{y}{6 + 8 + 10} = \frac{10 - x}{10}.$$

39. Distance to (4,0) plus distance to (0,-3) is 7. This is an ellipse with foci at (4,0) and (0,-3).

40. $Z = \sqrt{x^2 - 3} - 2$ must be as close to zero as possible and remain positive. This suggests $x^2 - 3 = 4$ or $x^2 = 7$. Since x is an integer, we might consider x = 2 or x = 3. But x = 2 makes Z negative. If x = 3, $Z = \sqrt{6} - 2$ and all larger values of x make Z larger than $\sqrt{6} - 2$. The answer is $\frac{1}{\sqrt{6} - 2} = \frac{\sqrt{6} + 2}{2}$.