EIGHTH ANNUAL

MICHIGAN MATHEMATICS PRIZE COMPETITION

sponsored by

The Michigan Section of the Mathematical Association of America,
Michigan Colleges and Universities, Professional Organizations, and Industries

PART I

NOVEMBER 5, 1964

INSTRUCTIONS
(to be read aloud to class by supervisor or proctor)

1. Do as many problems as you can in the 100 minutes allotted. When the proctor requests you to stop, please cease work immediately and turn in your answer card.

2. Essentially all of the problems require some figuring. Do not be hasty in your judgments. For each problem you should work out your ideas on scratch paper before selecting the answer.

3. Your score on the test will be the number right minus one-fourth the number wrong. You are advised to guess an answer only in those cases where you cannot determine the right answer but are able to eliminate some of the alternatives as impossible. If you have no ideas on a particular problem, you should leave it unanswered.

4. The average participant will have less than ten correct answers. To improve your score, be careful to complete all problems which you can do successfully before working on the other problems. Several questions, distributed throughout the test, can be answered by using only the first two years of high school mathematics.

5. Your answer sheet will be graded by machine. Please read and follow carefully the instructions printed on the sheet. Do not make calculations on the answer sheet.

6. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of this competition. If you have questions concerning the instructions, ask them now.

PART I

1. The perimeter of an equilateral triangle is 18. The area is

   (A) $3^{3/2}$ (B) $\frac{9}{4}\sqrt{3}$

   (C) $3^{5/2}$ (D) $4\sqrt{3}$

   (E) none of the above

2. Each angle of a regular polygon is 160°. How many sides has it?

   (F) 18 (G) 15

   (H) 24 (I) 36

   (J) none of the above

3. The solution set of $5 - 2x > x + 4$ is the set of all numbers x such that

   (A) $\frac{1}{3} < x$

   (B) $-\frac{1}{3} < x < \frac{1}{3}$

   (C) $\frac{1}{3} > x$

   (D) $x < 3$

   (E) none of the above

4. The inequality $|x| > \frac{1}{x}$ is not satisfied by any number such that

   (F) $|x| < 1$

   (G) $x^2 < 1$

   (H) $0 < x < 1$

   (I) $x < 1$

   (J) none of the above

5. Consider the six statements:

   (1) All women are good cooks.
   (2) No men are good cooks.
   (3) Some women are good cooks.
   (4) All men are bad cooks.
   (5) At least one man is not a good cook.
   (6) All men are good cooks.

   The statement which negates (6) is:

   (A) 1 (B) 3

   (C) 2 (D) 5

   (E) 4

6. Which of the following does not divide $x^4 - 2x^2y^2 + y^4$?

   (F) $x^2 - y^2$

   (G) $x^3 - xy^2 + yx^2 - y^3$

   (H) $x^2 - 2xy + y^2$

   (I) $x + y$

   (J) none of the above

7. The coefficients of the equation $21x + 102 = 10112$ are expressed in base three notation. The solution, in base three notation, is

   (A) 12

   (B) 110

   (C) $476\frac{2}{3}$

   (D) 11

   (E) none of the above

8. Sixty numbers have an average of 24. Two are discarded and the average of the remaining 58 is 25. Then the sum of the two discarded is

(F) 10 (G) 0

(H) -20 (I) -10

(J) none of the above

9. The equation
$x^2 + (1 - i)x - 2 - 2i = 0$
has for its roots

(A) $1 + i, -2$

(B) no roots

(C) $1 - i, -2 -2i$

(D) $\frac{1 - i \pm \sqrt{10 + 6i}}{2}$

(E) none of the above

10. If $x - y = \sqrt{3}$, $x + y = 2$, then $xy =$

(F) 1 (G) 4

(H) $\frac{1}{4}$ (I) $2\sqrt{3}$

(J) none of the above

11. If 5, x, y, 17 is an arithmetic progression, y =

(A) 9 (B) 13

(C) 15 (D) 11

(E) none of the above

12. x, y, and t are positive integers. $x + y = 15$. $xy = t^2$. How many possible values are there for t?

(F) 1 (G) 2

(H) 3 (I) 0

(J) none of the above

13. The graph of
$|x - 2| + |y + 3| = 2$

(A) circle (B) square

(C) ellipse (D) triangle

(E) none of the above

14. The ratio of the width of a rectangle to its length is 3 to 4. If the area is 1323, then the perimeter is

(F) 147 (G) 73.5

(H) 126 (I) $42\sqrt{21}$

(J) none of the above

15. Triangle ABC is isosceles with base BC. $\angle A$ is a 30° angle. Point P is within the triangle with $\angle PBC = \angle PAB$. Then the measure of $\angle APB$ is

(A) 150°

(B) 75°

(C) 120°

(D) not determined by this data

(E) none of the above

16. How many miles does a train go in 12 hours if it averages 40 mph when moving and makes n stops of m minutes each?

(F) $1440 - 2mn$

(G) $\frac{1440 - 2mn}{3}$

(H) $480 - 40mn$

(I) $\frac{480 - 2mn}{3}$

(J) none of the above

17. The repeating decimal .327327327 ... represents

(A) $\frac{327}{1000}$ (B) $\frac{109}{333}$

(C) $\frac{100}{327}$ (D) $\frac{91}{333}$

(E) none of the above

18. From a point within a triangle segments are drawn to the vertices. A necessary and sufficient condition that the three small triangles formed have equal areas is that the point be

(F) the intersection of the altitudes

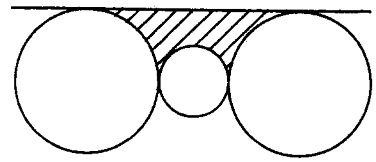
(G) the intersection of the medians

(H) the center of the inscribed circle

(I) the center of the circumscribed circle

(J) such that each angle is 120°

19.



The radius of each large circle is 6 and that of the small circle is 3. The three centers are collinear. The area of the shaded region below the tangent line is

(A) $54 - \frac{45}{2}\pi$

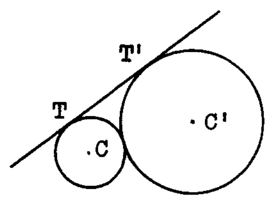
(B) $90 - \frac{45}{2}\pi$

(C) $108 - 18\pi$

(D) $108 - \frac{45}{2}\pi$

(E) none of the above

20.



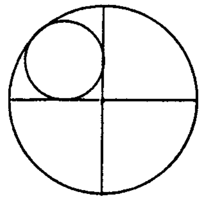
Points C and C' are centers of the circles, CT = 2, C'T' = 4. What is the length of the common tangent TT'?

(F) 4 (G) 6

(H) $4\sqrt{2}$ (I) $6\sqrt{2}$

(J) none of the above

21.



The large circle has radius 1, the two diameters shown are perpendicular to each other, and the small circle is tangent to both diameters and to the large circle. The radius of the small circle is

(A) $\sqrt{2} + 1$ (B) $\frac{1}{2}$

(C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2} - 1$

(E) none of the above

22. A 2-digit number is added to the number formed by reversing its digits. Then whatever 2-digit number was chosen, this sum is divisible by

(F) 9

(G) difference of the digit

(H) product of the digits

(I) the sum of the digits

(J) none of the above

23. In how many ways may 4 people be seated in a row if a particular two of them refuse to sit next to each other?

(A) 24 (B) 32

(C) 16 (D) 12

(E) none of the above

24. If x and y are real and $\frac{x}{y} = x + y$, then x may be any number satisfying

(F) $x > 0$ or $x < -4$

(G) $x > 0$ only

(H) $x > -4$

(I) $x < 0$ or $x > 4$

(J) none of the above

25. From the set of all couples having six children, 4 boys and 2 girls, one couple is to be chosen at random. What is the probability that the oldest and youngest children will both be boys?

(A) 2/5 (B) 3/5

(C) 1/2 (D) 4/9

(E) none of the above

26. In the expression $\frac{x + 1}{x - 1}$ each x is replaced by $\frac{x + 1}{x - 1}$. In this resulting expression, each x is replaced again by $\frac{x + 1}{x - 1}$. The value of this final expression for $x = \frac{1}{2}$ is

(F) 3

(G) -3

(H) 2

(I) -2

(J) none of the above

27. The number of integers less than 1000 which are not a square, not even, and not divisible by 5 is:

(A) 370 (B) 387

(C) 376 (D) 388

(E) none of the above

28. Accepting the approximations

$\log_{10}2 = .30103$, $\log_{10}3 = .47712$,

then $\log_5 9$ is approximately

(F) $\frac{.95424}{.69897}$ (G) $\frac{.95424}{.30103}$

(H) 1.90848 (I) 1.55630

(J) none of the above

29. If $a_0 = 1$, $a_1 = -1$, and

$a_n - a_{n-1}a_{n+1} = 1$, then a_3 is

(A) -3 (B) 3

(C) $\frac{1}{3}$ (D) 1

(E) -1

30. The remainder obtained when $x^{10} - 1$ is divided by $x^2 - a$ is

(F) $\frac{1}{a}$ (G) $a^5 - 1$

(H) $a^{10} - 1$ (I) $a - 1$

(J) none of the above

31. If the arithmetic mean of x and y is the product of $\sqrt{2}$ and the geometric mean of x and y, then $\frac{y}{x} =$

(A) $3 \pm 2\sqrt{2}$

(B) 1

(C) $3 \pm \sqrt{10}$

(D) $2 \pm 3\sqrt{3}$

(E) none of the above

32. The equation $\begin{vmatrix} 10^{x^2} & 10 \\ 10 & 10^x \end{vmatrix} = 0$

has as solutions

(F) $\frac{-1 \pm i\sqrt{3}}{2}$

(G) $\sqrt[3]{2}$

(H) 1,-2

(I) $\sqrt[3]{100}$

(J) none of the above

33. A cube is made by soldering twelve 3-inch wires properly at the vertices. A fly alights at one corner and walks along the edges. The greatest distance it could travel without retracing any edge is

(A) 27 inches (B) 24 inches

(C) 30 inches (D) 18 inches

(E) none of the above

34. In order for one root of $ax^2 + bx + c = 0$ to be three times the other, then a, b, c must satisfy

(F) $3b^2 = 4ac$ (G) $9b^2 = 16ac$

(H) $3b^2 = ac$ (I) $3b^2 = 16ac$

(J) none of the above

35. The roots of $ax^2 + bx + c$ are r and s. For the roots of $x^2 + px + q$ to be $\frac{1}{r}$ and $\frac{1}{s}$, p must be

(A) $\frac{-b}{c}$ (B) $\frac{b}{c}$

(C) $\frac{bc}{a^2}$ (D) $\frac{c}{b}$

(E) none of the above

36. A cylinder has radius 6 and height 4. The volume is increased twice as much when the radius is increased by x as it is when the height is increased by x. Then $x^2 + x$ equals

(F) 20 (G) 16

(H) 72 (I) 42

(J) none of the above

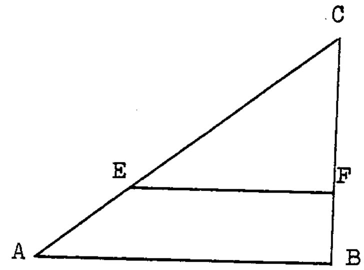
37. The expression $\frac{\cos\theta + \cos 3\theta}{\cos^2\theta - \sin^2\theta}$ is equivalent to

(A) $\cos 2\theta$ (B) $2\cos\theta$

(C) $\cot 2\theta$ (D) $\frac{1}{2}\cos\theta$

(E) none of the above

38.



Angle B is right. EF is parallel to AB. AB = 8, BC = 6, AE = x. The perimeter of ΔEFC is

(F) $12 - \frac{6x}{5}$ (G) $14 - \frac{7x}{5}$

(H) $24 - \frac{12x}{5}$ (I) $\frac{43}{12}(10 - x)$

(J) none of the above

39. The graph of the equation

$$\sqrt{(x-4)^2 + y^2} + \sqrt{x^2 + (y+3)^2} = 7$$

is

(A) an ellipse (B) a circle

(C) a parabola (D) a hyperbola

(E) none of the above

40. What is the largest value of $[\sqrt{x^2 - 3} - 2]^{-1}$ for integral x?

(F) $\frac{2 + \sqrt{13}}{9}$

(G) -1

(H) $\frac{2 + \sqrt{6}}{2}$

(I) no maximum value

(J) none of the above

The following Michigan companies and professional organizations have made contributions to the scholarship fund for this year's competition:

- Aeroquip Corporation, Jackson
- Burroughs Corporation, Detroit
- Clark Equipment Company, Battle Creek
- Lear Siegler, Incorporated, Grand Rapids
- The Michigan Council of Teachers of Mathematics
- Packaging Corporation of America, Filer City
- Thompson Ramo Wooldridge, Incorporated, Warren
- Whitman and Barnes, Plymouth

It is anticipated that other contributions will be received during the next few months. The names of these other companies will be reported in later announcements.

The Michigan Mathematics Prize Competition is an activity of the Michigan Section of the Mathematical Association of America.

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