FIFTH ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by

The Michigan Section of The Mathematical Association of America,
Michigan Colleges, Universities, Professional Organizations, and Industries

DECEMBER 13, 1962

PART I

INSTRUCTIONS
(To Be Read Aloud to Class by Supervisor or Proctor)

1. Do as many problems as you can in the 50 minutes allotted to Part I. When the proctor requests you to stop, please cease work immediately and turn in your answer card.

2. In general, no student is expected to finish all 40 problems in Part I. The first 20 questions can be answered on the basis of material normally found in the first two years of high school mathematics, while the last 20 questions may require the use of techniques which are taught only in the junior or senior year. This does not mean that the problems have been arranged according to levels of difficulty, since some relatively easy problems occur throughout Part I. You are expected to select those problems which you can do. If you can do 20 problems correctly, you will probably place in the top 3% of all contestants.

3. Your score on Part I will be the number right minus one-fifth the number wrong. If you don't know the right answer but can eliminate some alternatives as impossible, you are advised to guess an answer from the remaining alternatives.

4. Your card will be graded by machine. Please read and follow carefully the instructions printed on the card.

5. You may use blank paper for extra calculations, if it has been provided by your school. You may write on the examination booklet if you wish, but, in view of the electrical grading of the answer card, do not make calculations on the answer card.

6. Print your name, high school, and grade (9, 10, 11, or 12) in the space provided on your answer card. Write the number of the high school in the space provided; your supervisor will give you the number.

7. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of this competition. If you have questions concerning the instructions, ask them now.

PART I

1. $1 + \dfrac{1}{1 - \dfrac{1}{1+1}}$ is equal to

(A) -1 (B) $\frac{3}{2}$

(C) 3 (D) 2

(E) None of the above

2. If each number in a set of ten numbers is doubled, the average is

(A) multiplied by 5

(B) multiplied by 20

(C) increased by 20

(D) doubled

(E) none of the above

3. How many diagonals in a regular octagon?

(A) 20 (B) 40

(C) 24 (D) 56

(E) None of the above

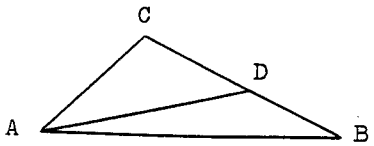
4. If 4 hens lay 4 eggs in 2 days, how many eggs should 6 hens lay in 3 days?

(A) 10 (B) 8

(C) 6 (D) 12

(E) None of the above

5.



$AC = CD$, $\angle CAB - \angle ABC = 30^\circ$. Then $\angle BAD$ has a measure of

(A) 15° (B) 30°

(C) 20° (D) $(22\frac{1}{2})^\circ$

(E) None of the above

6. The side of an equilateral triangle is 2. The area of the triangle is

(A) $2\sqrt{3}$ (B) $\sqrt{3}$

(C) $\dfrac{\sqrt{3}}{2}$ (D) $4\sqrt{3}$

(E) None of the above

7. Assume that the radius of the earth is 4,000 miles. At latitude 60° a person travels through 30 degrees of longitude. How many miles does he travel?

(A) $\dfrac{1000\sqrt{3}\,\pi}{3}$

(B) $\dfrac{2000\,\pi}{3}$

(C) $\dfrac{1000\,\pi}{3}$

(D) $\dfrac{2000\sqrt{3}\,\pi}{3}$

(E) None of the above

8. At a dance, 4 girls dance with a first boy, 5 girls with a second, 6 with a third, and so on until all girls dance with the last boy. Then, if there are b boys and g girls

(A) $b = g + 3$ (B) $g = b + 4$

(C) $g = 4b$ (D) $g = b + 3$

(E) None of the above

9. A student on vacation d days observed that

(1) there were 9 days when it rained in the morning or afternoon

(2) it rained both morning and afternoon on exactly one day

(3) when it was clear (that is, did not rain) in the morning, it rained in the afternoon

(4) there were 5 clear afternoons

Then d is

(A) 8 (B) 7

(C) 10 (D) 11

(E) None of the above

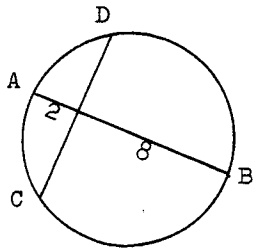
10. The g.c.d. of a and b is D. The l.c.m. of a and b is M. Then, if $M = \frac{ab}{2}$, D is

(A) 1 (B) $\frac{M}{a}$

(C) $\frac{M}{b}$ (D) 2

(E) None of the above

11.



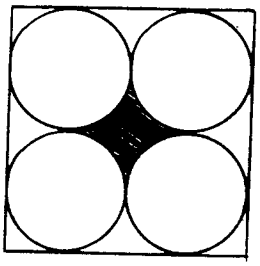
In the figure, AB is a diameter which bisects the chord CD at the point M. Given that AM = 2 and MB = 8, find the length of CD.

(A) 8 (B) $4\sqrt{3}$

(C) 10 (D) 4

(E) 5

12.



Four circles, each of radius 4, are drawn, as shown, in a square of side 16. The area of the shaded region is

(A) $32 - 16\pi$ (B) $16 - 16\pi$

(C) $16 - 4\pi$ (D) $16(4 - \pi)$

(E) None of the above

13. If $a = \frac{27}{13}$, then

$(3^a)(\sqrt[9]{3^a})(\sqrt[3]{3^a})$

(A) 27 (B) 9

(C) 3^{3a} (D) $9^{13/9}$

(E) None of the above

14. Consider the statement: "A triangle is isosceles if two of its sides are equal." In elementary geometry, this is

(A) a theorem

(B) a corollary

(C) a definition

(D) an axiom

(E) none of the above

15. From a point within a triangle, segments are drawn to the vertices. A necessary and sufficient condition that the three triangles formed have equal areas is that the point be:

(A) such that the three angles formed each have a measure of 120°

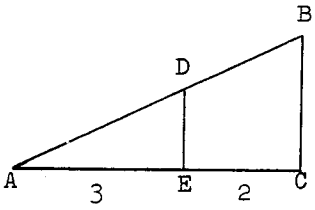
(B) the center of the inscribed circle

(C) the center of the circumscribed circle

(D) the intersection of the medians

(E) none of the above

16.



In triangle ABC, $\angle DEA = \angle BCA = 90^\circ$ and the area of ΔABC is 1. Given that AE = 3 and EC = 2, find the area of ΔADE.

(A) 3/5 (B) 1/3

(C) 9/10 (D) 4/9

(E) None of the above

17. The polynomial $2x + a$ is a factor of

(A) $2x^2 - a^2$

(B) $4xy + 2ay - 2bx - ab$

(C) $2xy + ay + xb + ab$

(D) $4x^2 + 2ax + a^2$

(E) None of the above

18. The largest integer n such that 5^n is a factor of $100! = (1)(2)(3)\cdots(99)(100)$ is

(A) 30 (B) 20

(C) 24 (D) 25

(E) None of the above

19. A two-digit number N is divisible by 5. When we subtract from this number the number formed by reversing the digits, we get a positive square. The number of possible values for N is

(A) 4 (B) 2

(C) 3 (D) 5

(E) None of the above

20. In how many ways may 5 persons be seated in a row if a particular one of them must be in an end seat?

(A) 48 (B) 24

(C) 120 (D) 60

(E) None of the above

21. The value of $(1 + i)^4$ is

(A) $8i - 4$ (B) -6

(C) -4 (D) 4

(E) None of the above

22. The smallest of the following numbers is

(A) $3\sqrt{7}$ (B) $2\sqrt{15}$

(C) $\sqrt{65}$ (D) $5\sqrt{3}$

(E) 8

23. If 1, x, y, $\frac{8}{27}$ form a geometric progression, then y is

(A) $\frac{16}{81}$ (B) $\frac{2}{3}$

(C) $\frac{9}{4}$ (D) $\frac{4}{9}$

(E) None of the above

24. The value of x for which $\log_x \frac{9}{16} = -2$ is

(A) $\frac{3}{4}$ (B) $\frac{4}{3}$

(C) $\frac{2}{3}$ (D) $-\frac{3}{4}$

(E) None of the above

25. The infinite repeating decimal 0.303030.... represents

(A) $\frac{33}{90}$ (B) $\frac{1}{3}$

(C) $\frac{3}{10}$ (D) $\frac{10}{33}$

(E) None of the above

26. If the roots of $(x-m)^2 + x = 0$ are equal, then m is

(A) 4 (B) $-\frac{1}{4}$

(C) $\frac{1}{4}$ (D) -4

(E) None of the above

27. If $a^x = b^{x+1}$, then x is

(A) $\frac{\log b}{\log(\frac{a}{b})}$ (B) $\frac{2 \log b}{\log a}$

(C) $\frac{\log a - \log b}{\log b}$

(D) $\log (b - \frac{a}{b})$

(E) None of the above

28. If s varies directly with x^2 and inversely with y, and if $s = 3$ when $x = 3$ and $y = 6$, then when $x = 2$ and $y = 2$, s is

(A) $\frac{1}{4}$ (B) 2

(C) 1 (D) 4

(E) None of the above

29. A and B can do a job together in 2 days, and B and C in 3 days. If A can do the job alone in 3 days, then A and C together can complete the job in

(A) 2 days (B) 1 day

(C) $2\frac{1}{2}$ days (D) $1\frac{3}{4}$ days

(E) None of the above

30. If $\begin{vmatrix} x & 1 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{vmatrix} = 0$, find the value of x

(A) -1 (B) 2

(C) 1 (D) $\frac{3}{2}$

(E) None of the above

31. If a die is rolled twice, the probability that at least one "6" is rolled is

(A) $\frac{10}{36}$ (B) $\frac{1}{3}$

(C) $\frac{1}{4}$ (D) $\frac{11}{36}$

(E) None of the above

32. If $\log 2 = 0.3010$, $\log 3 = 0.4771$, then the number of digits in $2^{12} \cdot 3^{10} \cdot 5^8$ is

(A) 14 (B) 13

(C) 30 (D) 10

(E) None of the above

33. If $3^{x+1} - 3^{x-1} = 24$, then $(2x)^{2x}$ is

(A) 27 (B) 256

(C) 625 (D) 1

(E) None of the above

34. If $4x^3 + 8x^2 - 3x + k$ is divisible by $x + 2$, then it is also divisible by

(A) $2x - 1$ (B) $4x^2 + 3$

(C) $4x^2 + 4x - 3$

(D) $4x^2 - 3$

(E) None of the above

35. If θ is an angle between 0° and 180° and $\cos \theta = -\frac{1}{3}$, then $\tan \theta =$

(A) $-2\sqrt{2}$ (B) $2\sqrt{2}$

(C) $\sqrt{10}$ (D) $-\sqrt{10}$

(E) None of the above

36. If $2 - \sqrt{3}$ is a root of $x^2 + 2(\sqrt{3} - 1)x + 3 - 2\sqrt{3} = 0$, then a second root is

(A) $\sqrt{3} - 2$ (B) $\sqrt{3}$

(C) $2 + \sqrt{3}$ (D) $-\sqrt{3}$

(E) None of the above

37. If arc tan a + arc tan b = 45°, then a + b =

(A) $\frac{1}{ab}$ (B) ab

(C) 1 - ab (D) 1 + ab

(E) None of the above

38. If $x < a < 0$, then

(A) $x^2 > ax$ but $ax < 0$

(B) $x^2 < ax < 0$

(C) $x^2 < a^2 < 0$

(D) $x^2 > ax > a^2$

(E) None of the above

39. Find all values of x for which $\left| \frac{5 - x}{3} \right| < 2$

(A) $1 < x < 11$

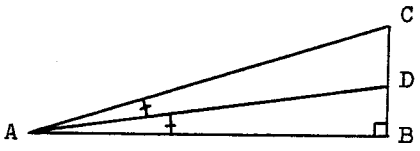
(B) $-1 < x < 11$

(C) $x < 11$

(D) $x > 11$

(E) $|x| < 6$

40.



If CD = 15, DB = 9, AD bisects $\angle A$, $\angle ABC = 90^\circ$, then AB has length

(A) 32 (B) 18

(C) 7 (D) 24

(E) None of the above