FOURTH ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by

Colleges, Universities, Professional and Industrial Organizations

In The State of Michigan

INSTRUCTIONS FOR PART I.

(To Be Read Aloud to Class by Supervisor or Proctor)

1. In general, the contestant is not expected to finish all problems in Part I. Do as many problems as you can in the fifty minutes given to Part I. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

2. Your answer sheet will be graded by electrical machine, so follow carefully these instructions:

   a) Fill in the space on the answer sheet only with the special pencil provided.

   b) Incorrect answers will be penalized, so do not guess; leave the space blank if you cannot solve a particular problem.

   c) Fill in at most one space on the answer sheet; there is only one correct answer, so two or more answers to one question will produce extra penalties.

   d) Be particularly careful not to rest the point of the special pencil on the answer sheet; any mark made by the pencil may conceivably carry an electrical current and give you an undeserved penalty.

   e) If you wish to change an answer to a particular problem, do not cross out your first answer, but rather erase it completely. Remember that the electrical current to be used in scoring will not understand your intentions.

3. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of this competition. If you have questions concerning the instructions, ask them now, before the Part I is distributed to you.

4. You may use blank paper for extra calculations, if it has been provided by your school. You may write on the examination booklet if you wish, but, in view of the electrical grading of the answer sheet, do not make calculations on the answer sheet.

5. Print your name, high school, and grade in the space provided on your answer sheet. After the high school, write the number of the high school in the space provided; your supervisor will give you the number.

March 2, 1961

Part I

1. If the average cost of an article is W, then the cost of N articles is

   (A) NW (C) W/N

   (B) N/W (D) W-N

   (E) None of the above.

2. $(x+y)^{-1} + (x-y)^{-1} =$

   (A) $(x^2-y^2)(2x)^{-1}$ (C) $(x^2-y^2)^{-1}$

   (B) $(2x)^{-1}$ (D) $2x(x^2-y^2)^{-1}$

   (E) None of the above.

3. The value of $3x^0 + \left(\frac{4}{x}\right)^{-2} + \sqrt[3]{x^2}$, when $x = 8$, is

   (A) 7 (C) 9

   (B) 7 1/2 (D) 11

   (E) None of the above.

4. Which of the following numbers is closest to $\frac{1}{5}\left(\frac{3}{4} + \frac{7}{8}\right)^{-1}$?

   (A) $\frac{1}{6}$ (C) $\frac{1}{8}$

   (B) $\frac{1}{7}$ (D) $\frac{1}{9}$

   (E) $\frac{1}{10}$

5. If $\left(x^2 - \frac{1}{x^3}\right)^8$ is arranged as a sum of powers of x, which of the following powers of x does not appear with non-zero coefficient?

   (A) x^{11} (C) x^3

   (B) x^{-4} (D) x^{-19}

   (E) x

6. If 1, x, y, 27 form a geometric progression, x=

   (A) 9 (C) $\sqrt{3}$

   (B) 3 (D) $\sqrt[3]{9}$

   (E) None of the above.

7. If 2, x, y, 11 form an arithmetic progression, x=

   (A) 8 (C) $\frac{13}{4}$

   (B) 9/4 (D) 5

   (E) None of the above.

8. $\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} =$

   (A) $\frac{1}{2}\tan\theta$ (C) $\tan\theta/2$

   (B) $\cot\theta/2$ (D) $\frac{1}{2}\cot\theta$

   (E) None of the above.

Part I

9. In how many different ways can a committee of three be chosen from a group of five people?

(A) 15 (C) 60

(B) 10 (D) 125

(E) None of the above.

10. Which of the following numbers is real?

(A) $(i)^5$ (C) $(1+i)^2$

(B) $\frac{(3+i)}{(3-i)}$ (D) $(1-i)^2$

(E) None of the above.

11. The maximal value of $3 \sin \frac{1}{2}\theta$ is

(A) 3/2 (C) 3

(B) 1 (D) 1/2

(E) None of the above.

12. What is the smallest value assumed by $x^2 + 8x$, for x real?

(A) 0 (C) -124

(B) -16 (D) 9

(E) None of the above.

13. In the figure, the area of triangle ADE equals the area of the trapezoid EDBC. If BC = 12, then DE =

(A) 6 (C) $9\sqrt{3}$

(B) 36 (D) $6\sqrt{2}$

(E) None of the above.

14. A rhombus has a side of length 12 and an angle of 60°. Its area is

(A) 144 (C) $36\sqrt{3}$

(B) $72\sqrt{3}$ (D) 108

(E) None of the above.

15. The value of $\tan 1050^\circ$ is

(A) $\sqrt{3}$ (C) $1/\sqrt{3}$

(B) $-\sqrt{3}$ (D) $-1/\sqrt{3}$

(E) None of the above.

16. $\cos 3\theta =$

(A) $3\cos^3\theta + 4\cos\theta$ (C) $4\cos^3\theta - 3\cos\theta$

(B) $4\sin^3\theta - 3\sin\theta$ (D) $3\sin^3\theta + 4\sin\theta$

(E) None of the above.

17. If θ is an angle between 0° and 180° and $\tan\theta = -2$, then $\cos\theta =$

(A) $-1/\sqrt{5}$ (C) $2\sqrt{5}$

(B) $-\sqrt{5}$ (D) $2/\sqrt{5}$

(E) None of the above.

18. If a and b are positive numbers and $a^x = b^{1-x}$, then $x =$

(A) $\frac{\log b}{\log ab}$ (C) $\log b - \log ab$

(B) $\frac{1}{\log a}$ (D) $\log a - \log b$

(E) None of the above.

19. x varies inversely with y^2 and directly with $\sqrt{w}$, and $x = 16$ when $y=2$ and $w=4$. If $x=8$ and $y=4$, then $w=$

(A) $\frac{1}{\sqrt{2}}$ (C) 4

(B) $\frac{1}{4}$ (D) $\sqrt{2}$

(E) None of the above.

20. Which of these answers satisfies the inequality

$$\frac{x^2-4}{x^2+1} > 0$$

(A) $x < 2$ (C) $x > 2$ or $x < -2$

(B) $x > -2$ (D) $-2 < x < 2$

(E) None of the above.

21. Assuming that the radius of the earth is 3000 miles, how far does a person travel on the surface if, at latitude 30° he moves 10° of longitude?

(A) $\frac{1000\pi}{3}$ (C) $\frac{1500\pi}{\sqrt{3}}$

(B) $\frac{1000\pi}{3\sqrt{3}}$ (D) $\frac{500\pi}{3}$

(E) None of the above.

22. A circle of radius 10 is circumscribed about a triangle ABC. If $AB = BC = 10$, then the area of the triangle is

(A) 50 (C) $25\sqrt{2}$

(B) $25\sqrt{3}$ (D) 40

(E) None of the above.

23. A circle of radius 2 is inscribed in a regular hexagon. The area of the hexagon is

(A) $2\sqrt{3}$ (C) $3\sqrt{3}$

(B) 6 (D) $6\sqrt{2}$

(E) None of the above.

Part I

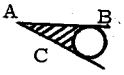
24. The altitude to the hypotenuse of a right triangle divides the hypotenuse into segments of lengths 2 and 8. The length of the altitude is

(A) $\sqrt{60}$ (C) 5

(B) $2\sqrt{5}$ (D) 4

(E) None of the above.

25. In the figure, lines AB and AC are tangents to the circle of radius 6 and $\angle BAC = 60^\circ$. The area of the shaded region is



(A) $36\sqrt{3} - 12\pi$ (C) $10\sqrt{6} + 2\sqrt{3} - 12\pi$

(B) $12(\sqrt{3} - \pi)$ (D) $54 - 9\pi$

(E) None of the above.

26. Let p(x) be a polynomial of lowest possible degree which has whole numbers as coefficients and $i, \sqrt{2}$, 1 as roots. Which of the following is not a factor of p(x)?

(A) $x^2 - 2$ (C) $x - 1$

(B) $x^2 + 1$ (D) 1

(E) None of the above.

27. Let $p(x) = 2 + a_1x + \ldots + a_{n-1}x^{n-1} + 6x^n$. Which of the following numbers cannot possibly be a root of p(x)?

(A) 2/3 (C) 1

(B) - 1/6 (D) 3/2

(E) -2

28. The two roots, r_1 and r_2, of the equation $x^2 + px + q = 0$ satisfy the linear equations $r_1 - 2r_2 = 2$, $2r_1 - 3r_2 = 5$. The value of p is

(A) -5 (C) 4

(B) 5 (D) 1/4

(E) None of the above.

29. Find a value of k, in the list below, for which the equation $k^2x^2 + x + 4 = 0$ has more than one real root.

(A) 1/4 (C) 1

(B) - 1/4 (D) 0

(E) None of the above

Part I

30. Two circles have radii a, b. The distance between their centers is d. They intersect at P and Q. The distance from P to Q is

(A) $\dfrac{\left[d^2 - (a - b)^2\right]^{1/2}\left[(a+b)^2 - d^2\right]^{1/2}}{d}$

(B) $(a^2 + b^2 - d^2)^{1/2}$

(C) $\left[(a - b)(a + b)\right]^{1/2}$

(D) $\dfrac{a + b + d}{3}$

(E) None of the above.

31. Find a nonzero root of the equation

$$\begin{vmatrix} x^3 & 0 & 1 \\ x^2 - 1 & 1 & x^2 \\ x + 2 & -2 & x \end{vmatrix} = 0$$

(A) 1 (C) -2

(B) 1/2 (D) - 1/2

(E) None of the above.

32. If a square of side 12 is rotated in its plane about a point on one side 3 units from a vertex, the area of the figure swept out is

(A) 2500 (C) 144π

(B) 225π (D) 81π

(E) None of these.

33. A circular segment is cut from a regular pentagon, with one vertex as the center and a side of the pentagon as the radius of the circle. If the area of the segment is 17 then the length of each side of the pentagon is

(A) $\sqrt{17}$ (C) 5π / 17

(B) $\frac{1}{2}\sqrt{17}$ (D) $\sqrt{85/\pi}$

(E) None of the above.

34. A plane cuts a corner off a cube so that the lengths of the three line segments cut from the edges of the cube all equal 1. The volume of the corner cut off is

(A) $\sqrt{3}/8$ (C) 1/6

(B) 1/2 (D) 1/3

(E) None of the above.

35. At how many minutes after 8 o'clock does the minute hand lie on top of the hour hand?

(A) 43 7/11 (C) 44 1/30

(B) 41 1/12 (D) 42

(E) None of the above.

36. A man engages in a shooting contest. Each time he hits a target he receives 10 cents, while each time he misses he pays 5 cents. If, after 20 shots he has lost 10 cents, how many times has he hit the target?

(A) 6 (C) 10

(B) 2 (D) 4

(E) None of the above.

37. A man makes a canoe trip 12 miles upstream and returns to the starting point. His total travel time is 9 hours. If the man can paddle 3 miles per hour in still water, how fast is the stream flowing?

(A) 1 m. p. h. (C) 4/3 m. p. h.

(B) 8/9 m. p. h. (D) $2\sqrt{5}$ m. p. h.

(E) None of the above.

38. A parachutist observes, as he jumps from his balloon which is 2700 yards above the ground, that a point P on the ground subtends an angle, θ, from the vertical. As his parachute opens, 1200 yards above the ground, he notes that the angle subtended by P has increased by $\frac{90^{\circ} - \theta}{2}$. Assuming his descent is vertical, how far is he from P when he lands?

(A) 3600 yards (C) 900 yards

(B) 2700 yards (D) $300\sqrt{41}$ yards

(E) None of the above

39. If $x - 1/x = 2i \sin\theta$, then $x^2 - \frac{1}{x^2} =$

(A) $-4 \sin^2\theta$ (C) $4i \sin 2\theta$

(B) $-4 \sin 2\theta$ (D) $-4x^2 \cos 2\theta$

(E) None of the above.

40. If a(n, r) and s(n, r) stand, respectively for $\frac{1-(1+r)^{-n}}{r}$ and $\frac{(1+r)^n - 1}{r}$, then $\frac{1}{a(n, r)} - \frac{1}{s(n, r)} =$

(A) r (C) r^n

(B) $(1 + r)^n$ (D) 0

(E) None of the above.