PART I

FIRST ANNUAL MICHIGAN MATHEMATICS PRIZE COMPETITION

Sponsored by

Colleges, Universities, Professional and Industrial Organizations

In The State of Michigan

INSTRUCTIONS FOR PART I.

(To Be Read Aloud To Class By Supervisor or Proctor)

1. In general, the contestant is not expected to finish all problems in Part I. Do as many problems as you can in the fifty minutes given to Part I. When the proctor requests you to stop, please cease work immediately and turn in your answer sheet.

2. Your answer sheet will be graded by electrical machine, so follow carefully these instructions:

   a. Fill in the space on the answer sheet only with the special pencil provided.

   b. Incorrect answers will be penalized, so do not guess; leave the space blank if you cannot solve a particular problem.

   c. Fill in at most one space on the answer sheet; there is only one correct answer, so two or more answers to one question will produce extra penalties.

   d. Be particularly careful not to rest the point of the special pencil on the answer sheet; any mark made by the pencil may conceivably carry an electrical current and give you an undeserved penalty.

   e. If you wish to change an answer to a particular problem, do not cross out your first answer, but rather erase it completely. Remember that the electrical current to be used in scoring will not understand your intentions.

3. The person supervising this test is not permitted to explain to you the meaning of any question, so do not request your supervisor to break the rules of this competition. If you have questions concerning the instructions, ask them now, before the Part I is distributed to you.

4. You may use blank paper for extra calculations, if it has been provided by your school. You may write on the examination booklet if you wish, but, in view of the electrical grading of the answer sheet, do not make calculations on the answer sheet.

5. Print your name and high school in the space provided on your answer sheet. After the high school, write the number of the high school in the space provided; your supervisor will give you the number.

March 27, 1958

PART I

1. The expression

$5x - (x - 6y) - 2y - (3x + 4y)$ equals

(A) $x - 12y$ (B) $9x - 4y$
(C) y (D) $10y$
(E) x

2. If the cost of P articles is A, the average cost per article is

(A) P A (B) A/P
(C) P/A (D) P - A
(E) None of the above.

3. If $4ax - 3 = 15 - 2ax$, x equals

(A) $3/a$ (B) $3a$
(C) $4/a$ (D) 18
(E) None of the above.

4. Each side of an equilateral triangle is 10. The altitude of the triangle is

(A) $10\sqrt{3}$ (B) $10/\sqrt{3}$
(C) $5\sqrt{3}$ (D) $\sqrt{10}$
(E) None of the above.

5. In the right triangle ABC, CD is the altitude drawn to the hypotenuse AB. If AD = 5 and DB = 14, then CD is

(A) $\sqrt{140}$ (B) 70
(C) 14/5 (D) $\sqrt{171}$
(E) None of the above.

6. The quantity $(3x - 2a)$ is a factor of

(A) $9x^2 + 4a^2$ (B) $3x^2 - 2a^2$
(C) $3xy + 9bx - 2ay - 6ab$
(D) $9x^2 - 6ax + 4a^2$
(E) None of the above.

7. S varies directly with x and inversely with y^2 and S = 9, when x = 12 and y = 2. When x = 18 and y = 3, S equals

(A) 6 (B) 2
(C) 4 (D) 12
(E) None of the above.

8. The value of $(-1/2 + i\sqrt{3}/2)^3$ is

(A) $-1/2 - i\sqrt{3}/2$ (B) $-1/2 + i\sqrt{3}/2$
(C) 1 (D) $1 + i$
(E) -1

9. Two radii OA and OB of a circle intercept an arc $\overset{\frown}{AB}$ of length 1. If OA = 2, the area of the circular sector OAB is

(A) 4π (B) $2\pi/15$
(C) 2 (D) 1
(E) None of the above.

10. The corresponding sides of two similar triangles are in the ratio 2 to 3. If the area of the smaller triangle is 12, the area of the larger is

(A) 24 (B) 27
(C) 18 (D) 8
(E) None of the above.

11. The value of x in the equation $\log_x(27/8) = -3$ is

(A) 3 (B) 3/2
(C) 2 (D) 2/3
(E) 10

12. The complex fraction

$(a/2 + b/3) \div (b/a + 3/2)$ when simplified equals

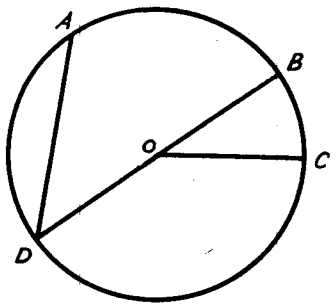
(A) ab (B) b/2
(C) ab/3 (D) ab/2
(E) None of the above.

13. Two chords intersect inside a circle. One chord is divided into segments 6 and 8 units long by the second chord. If one segment of the second chord is 4, the length of the other segment is

(A) 10 (B) 12
(C) 16 (D) 18
(E) None of the above.

14. In the figure, O is the center of a circle of radius 3. If $\angle ADB = 44^\circ$ and $\angle BOC = 32^\circ$, the length of the arc $\overset{\frown}{ABC}$ is

(A) 3π (B) $3\pi \cos 76^\circ$
(C) $3\pi/2$ (D) $\pi/2 \cos 76^\circ$
(E) None of the above.



15. The quantity $-2^3(1/2)^0(-8)^{-\frac{2}{3}}$ equals

(A) 4 (B) 2
(C) -4 (D) -2
(E) 0

16. A circle is inscribed in a right triangle having sides 3, 4, and 5. The area of the inscribed circle is

(A) $25\pi/4$ (B) 4π
(C) π (D) 9π
(E) None of the above.

17. The fourth term of the arithmetic progression $(3 - \sqrt{2})$, 2,---- is

(A) $3 + 2\sqrt{2}$ (B) $2\sqrt{2}$
(C) $3 - 2\sqrt{2}$ (D) $3\sqrt{2}$
(E) $1 + \sqrt{2}$

18. A regular octagon is inscribed in a circle of radius 4. The area of the octagon is

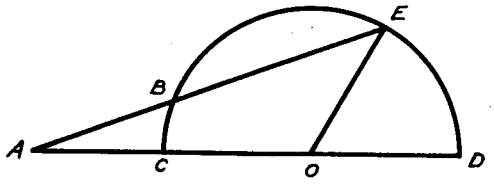
(A) 16π (B) $128\sqrt{2}$
(C) $32\sqrt{2}$ (D) $8\sqrt{2}$
(E) None of the above.

19. The difference between two numbers is 2. Their product is 84 greater than the square of the smaller number. The sum of the numbers is

(A) 21 (B) 42
(C) 84 (D) 86
(E) 164

20. In the figure, CD is the diameter of a semicircle $\overset{\frown}{CBED}$ with center O, and AB = OD. If $\angle EOD = 60^\circ$, then $\angle BAC$ is

(A) 15° (B) 20°
(C) 30° (D) 45°
(E) None of the above.

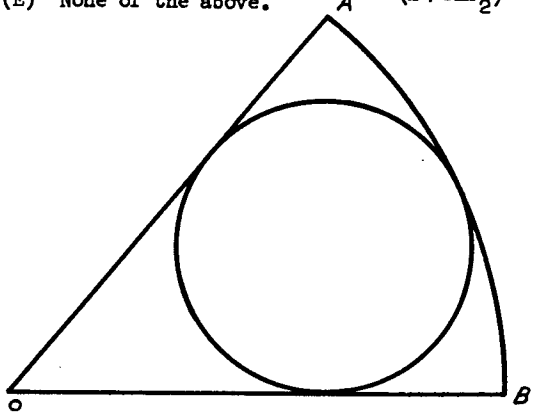


21. The roots of $(x + m)^2 + 2x = 0$ are equal if m is

(A) 1 (B) -1
(C) -1/2 (D) -2
(E) None of the above.

22. A circular sector OAB has a radius R and a central angle θ. The area of the inscribed circle is

(A) $\pi R^2 \cos^2 \frac{\theta}{2}$ (B) $\pi R^2 \sin^2 \frac{\theta}{2}$
(C) $\pi R^2 (1 - \cos \frac{\theta}{2})^2$ (D) $\pi R^2 \frac{(1 - \cos\frac{\theta}{2})}{(1 + \sin\frac{\theta}{2})^2}$
(E) None of the above.



23. If thirty five men can do a job in fifteen 8-hour days, the number of men required to do the same job in twenty four 7-hour days is

(A) 15 (B) 25
(C) 21 (D) 35
(E) None of the above.

24. A man flies due east from a point on the earth having latitude 60°N, longitude 140°W to a point with latitude 60°N, longitude 110°W. Assuming the radius of the earth to be 4,000 miles, the man has flown approximately

(A) 333π miles (B) $1{,}667\pi$ miles
(C) 667π miles (D) $667\pi\sqrt{3}$ miles
(E) None of the above.

25. The geometric mean between 2/3 and 6/25 is

(A) 34/75 (B) 2/5
(C) 2/25 (D) 3/5
(E) None of the above.

26. The largest possible value of $7 \sin 3x + 24 \cos 3x$ is

(A) 31 (B) 17
(C) 31/3 (D) 25
(E) None of the above.

27. A crew rows four miles downstream and back the same distance in one hour. If the stream flows at three miles per hour, the crew's rate of rowing in still water is

(A) 4 mph (B) -1 mph
(C) 8 mph (D) 12 mph
(E) 9 mph

28. If $\arctan x + \arctan k = 45^\circ$, then x is

(A) $1 + k$ (B) $1 - k$
(C) $1 - k^2$ (D) $\frac{1-k}{1+k}$
(E) None of the above.

29. The expression $\frac{x}{x^2 - 1}$ has no meaning when

(A) $x = 0$ (B) $-1 < x < 1$
(C) $x = \pm i$ (D) $x = \pm 1$
(E) None of the above.

30. If a, b, c are the sides opposite the angles A, B, C, respectively, in a triangle ABC, the expression $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ is equal to

(A) $2(a^2 + b^2 + c^2)$ (B) 1
(C) $\frac{a^2 + b^2 + c^2}{2abc}$
(D) $\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c}$
(E) None of the above.

31. Three roots of the equation $2x^4 - 3x^3 + 6x^2 - 12x - 8 = 0$ are $2i$, $-2i$ and 2. The fourth root is

(A) 1 (B) -1
(C) 1/2 (D) -1/2
(E) None of the above.

32. A circle of radius a is rotated about a line not intersecting the circle to form a solid called a torus. If the distance of the line to the center of the circle is b, the volume of the torus is

(A) πab^2 (B) $\pi a^2 b$
(C) $\pi a(b - a)^2$ (D) $2\pi^2 a^2 b$
(E) None of the above.

33. The number of real solutions to the system of equations $16x^2 + 25y^2 = 360$ and $xy = 1$ is

(A) 0 (B) 1
(C) 2 (D) 3
(E) 4

PART I

34. If $\log_{10}2 = .30103$ and $\log_{10}3 = .47712$, then $\log_{10}0.18$ is

(A) 9.25527-10 (B) 1.25527

(C) 9.77815-10 (D) .17609

(E) 9.17609-10

35. The roots of $x^3 + x^2 - 2x - 2 = 0$ are

(A) one real, two complex

(B) all complex

(C) all rational

(D) one rational, two irrational

(E) all irrational

36. A triangle OAB consists of two radii OA and OB of a circle with center O and radius r. The maximum area which the triangle can have is

(A) $r^2\sqrt{2}$ (B) $\frac{r^2\sqrt{2}}{4}$

(C) $\frac{\pi r^2}{4}$ (D) $r^2/2$

(E) None of the above.

37. The third term in the expansion of $(x - 2y)^9$ is

(A) $8x^4y^5$ (B) $144x^7y^2$

(C) $-8x^5y^4$ (D) $-72x^2y^7$

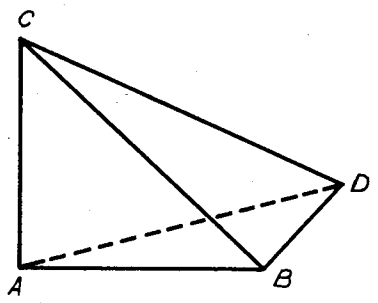
(E) $72x^6y^3$

38. In the three dimensional figure $\angle BAC = \angle DAC = \angle DBA = \angle DBC = 90^\circ$. The tangent of angle ADC is

(A) tan ABC sin ADB (B) cot ADB cos ABC

(C) sec ADB cos CDB (D) tan ADB cot CDB

(E) None of the above.



39. If $x = 4$ is a solution of the equation $a(x - 1)^4 + b(x - 1)^2 + c = 0$, then a second solution is

(A) $x = -4$ (B) $x = -1$

(C) $x = -2$ (D) $x = 2$

(E) $x = 1$

40. If $\cos\theta \neq 1$, then the sum of the series $1 + \cos\theta + \cos 2\theta + \cos 3\theta + \ldots$ is

(A) $1 + \sqrt{\cos\theta}$ (B) $2\cos^2\frac{\theta}{2}$

(C) $\frac{\sin\theta}{1 - \cos\theta}$ (D) ∞

(E) None of the above.