Theorems, Proofs, and Logic

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Fall 2009
Basics

- A *statement* is an assertion that can be classified as either true or false (but not both).
- The *existential quantifier* ($\exists$) asserts that something exists.
- The *universal quantifier* ($\forall$) claims that something holds for all members of a class.
- The *negation* of a statement is its opposite. In particular, the negation of a true statement is false and vice versa. The shorthand can be $\neg$, $\sim$, or the word “not”.
- *Unique* means “exactly one”. Shorthand is !
### Truth Tables

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A$ and $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
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<tr>
<td>$T$</td>
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<tr>
<td>$F$</td>
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</tr>
</tbody>
</table>

and:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A$ or $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
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<tr>
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<td>$F$</td>
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</table>

"Or" is inclusive.

"Not" negates the statements and swaps "and" for "or" and $\exists$ for $\forall$. 

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Theorems, Proofs, and Logic
A conditional statement is an “if-then” ($A \implies B$). $A$ is called the hypothesis, and $B$ is called the conclusion. A conditional is true if the conclusion holds whenever the hypothesis does.

A theorem is a (conditional) statement that has been proved true.

The converse of the statement “$A \implies B$” is “$B \implies A$”. The truth value of the converse is completely unrelated to the truth value of the original statement.

The contrapositive of the statement “$A \implies B$” is “$\sim B \implies \sim A$”. The contrapositive is logically equivalent to the original statement.

The negation of “$A \implies B$” is “$A$ and $\sim B$”.

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Theorems, Proofs, and Logic
A proof is a sequence of small, justified, logical steps that lead from hypothesis to conclusion.

For now, we’ll put the justification (reason) for each step in parenthesis after that step.

Valid justifications are:

- hypothesis
- axiom
- previous theorem
- definition
- earlier step in the proof
- rule of logic
Examples

Do some examples.
Indirect Proofs

- Starting with the hypothesis and arguing to the conclusion is a *direct proof*.
- Starting with both the hypothesis and the negation of the conclusion and deducing a logical contradiction is an *indirect proof*, a proof by contradiction, or a *reductio ad absurdum* (RAA).
- The negation of the conclusion is called the *RAA hypothesis*.
- Try to avoid indirect proofs if possible.